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A normal number theorem for Brownian motion

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(Recommended by Josselin Garnier)

Abstract. We prove that we can generate "a lot" of random normal numbers (in the sense of full Hausdorff dimension), which are outside the scope of Borel's Normal Number Theorem (in the sense of zero Lebesgue measure). The ingredient is to run Brownian motion over a specific Cantor-like time-set. These are closely related to an equidistributed property of some dynamical orbit with random inputs, which itself is new and significant.

1. Main Result

In this short article, we prove that, for Brownian motions living in the unit interval and in the two-dimensional torus, with probability 1, there are associated sets of normal numbers that are of full Hausdorff dimension and of zero Lebesgue measure. We start with the one-dimensional case. For this, we use a self-similar *two-fourths* Cantor set, denoted as $C_{4,2}$, by which we mean the limit set of the iterated scheme to start with dividing [0,1] into 4 equal subintervals and removing the second and the third ones, and then to proceed the iteration in a self-similar way. The $C_{4,2}$ is a Hausdorff β-set, with $\beta = \ln 2/\ln 4 = 1/2$. The $h_\beta(\cdot)$ denotes the Hausdorff β-dimensional measure; the $\dim_H(\cdot)$ denotes the Hausdorff dimension, and the $Leb_d(\cdot)$ denotes the Lebesgue measure in the Euclidean space $\mathbb{R}^d$. For more terminologies, see those explained below.

Theorem 1.1. *Let $\bar{B}(t), t \in [0,1]$, be the Brownian motion in the unit interval of $\mathbb{R}$, obtained by the modulus 1 values of Brownian motion $B(t)$ in the real line. With probability 1, for $h_{1/2}$-almost every $t \in C_{4,2} \subset [0,1]$, $\bar{B}(t)$ is normal in every given base b. Moreover, let $N_{4,2}$ denote the (random) set of all those $t \in C_{4,2}$ such that $\bar{B}(t)$ is a normal number; then, with probability 1, $\dim_H(\bar{B}(N_{4,2})) = 1$, while $Leb_1(\bar{B}(N_{4,2})) = 0$.*

A number $x \in [0,1]$ is said to be *(simply) normal in an integer base* $b > 1$ if the b-digits expansion of $x := \sum_{n=1}^{\infty} x_n b^{-n}$, with $0 \le x_n \le b-1$ and $x_n < b-1$ infinitely often (which ensure that the expansion is infinite), has the asymptotic frequency $1/b$; that is, for each $d : 0 \le d \le b-1$,

$$\lim_{n\to\infty} \frac{\sharp\{k \le n : x_k = d\}}{n} = \frac{1}{b},$$

where $\sharp$ denotes cardinality. We refer to Harman [5] and Khoshnevisan [8] for surveys on normal numbers. We should remark that, in this short article, our main results are normality in any given base, and thus the elaborated notions for *simple/entire/absolute* normality in these surveys (which are all due to Émile Borel) are not used, for the convenience of reading. We refer to Mörters and Peres [10] for a monograph on Brownian motions.

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Remark 1.2. We may call that the random point $\bar{B}(t)$ appearing in the conclusion of Theorem 1.1 a *Brownian normal number*. Then, the $\dim_H(\bar{B}(N_{4,2})) = 1$ result of Theorem 1.1 tells us that such random normal numbers are 'quite a lot'; whilst the set $\bar{B}(N_{4,2})$ is of Lebesgue measure 0, and thus such Brownian normal numbers are *outside the scope* of the classical Borel Normal Number Theorem (which says that Lebesgue almost every number in $[0,1]$ is normal in any given base b).

Remark 1.3. In Theorem 1.1, we use $C_{4,2}$ as our "fractal time-set"; we can consider other Cantor-like set as our time-set; see Section 2 of Laba [9] for constructions of general Cantor sets (we can choose here t, N to have $\ln t / \ln N = 1/2$, say, $t = 3$ and $N = 9$); that article discusses topics related to Fourier decay of fractal measures. We should mention that, firstly the self-similarity construction of $C_{4,2}$ is crucial, and secondly the more familiar middle-thirds Cantor set (start with dividing $[0,1]$ into three equal ones and removing the middle one) does not fit our need (see Remark 1.6). Therefore, the $C_{4,2}$ is an exact choice for Theorem 1.1.

We state the two-dimensional version as follows.

Theorem 1.4. *Let $\bar{B}(t), t \in [0,1]$, be the Brownian motion living in the two-dimensional torus $\mathbb{T}^2$. With probability 1, the random set of planar Brownian normal numbers in $[0,1]^2$ (both the x- and the y- components are normal) is of Hausdorff dimension 2 and of planar Lebesgue measure 0.*

The proofs of Theorems 1.1 and 1.4 have three parts: normality, full Hausdorff dimension, and zero Lebesgue measure. The normality can be derived as a consequence of the following more general equidistribution result, which is new and significant as to our knowledge.

Theorem 1.5. *Given $N \geq 1$, $d \geq 1$, and $H : 0 < H < 1$. Let $\bar{X}(t)$ be an N-parameter fractional Brownian motion in the d-dimensional torus $\mathbb{T}^d$, with Hurst index H, and let $T : \mathbb{T}^d \to \mathbb{T}^d$ be an expanding endomorphism. Given a compact $E \subset \mathbb{R}^N$ which is a Hausdorff β-set, with $\beta \leq Hd$. With probability 1, for h_β-almost every $t \in E$, the sequence $x_n := T^n(\bar{X}(t)), n = 0,1,2,\ldots$, is an equidistributed sequence in $\mathbb{T}^d$; that is, for each ball $B \subset \mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{\sharp\{k \leq n : x_k \in B\}}{n} = \frac{\mu(B \cap \mathbb{T}^d)}{\mu(\mathbb{T}^d)},$$

where μ denotes the Haar measure for $\mathbb{T}^d$, identified as the Lebesgue measure mod 1.

Remark 1.6. Theorem 1.5 does not hold in case $\beta > Hd$ (see the paragraph in Section 2 located immediately above the proof of Theorem 1.1); in particular, Theorem 1.5 can only directly apply to $\beta \leq 1/2$ for the linear Brownian path (for which $N = d = 1$ and $H = 1/2$). However, in this aspect, we can have a more general statement via indications by one referee; see the context of the proof of Theorem 1.1. Moreover, Theorem 1.5 can be proceeded for (N,d) Gaussian fields more general than fBm, indeed it works for a Gaussian field with stationary increments for which the spectral measure satisfies certain asymptotic index conditions; see Shieh and Xiao [11] for a detailed description of such fields.

In applications of Theorem 1.5 to Theorems 1.1 and 1.4, we need only the 1-parameter case; the the following 2-parameter case is of its own interest and we list it separately.

Corollary 1.7. *Let $\bar{B}(s,t)$ be the 2-parameter Lévy's Brownian motion living in the 3-dimensional torus $\mathbb{T}^3$, and let $T : \mathbb{T}^3 \to \mathbb{T}^3$ be an expanding endomorphism. With probability 1, for $(h_{1/2} \times Leb_1)$-almost every $(s,t) \in C_{4,2} \times [0,2\pi]$, the sequence $T^n(\bar{B}(s,t)), n = 0,1,2,\ldots$, is an equidistributed sequence in $\mathbb{T}^3$.*

Remark 1.8. The product set $C_{4,2} \times [0,2\pi]$ is called a Cantor target in Chapter 7 of Falconer [3], as an example of product fractals in $\mathbb{R}^2$ (we replace his middle-thirds Cantor set there by $C_{4,2}$ to meet our need).

We explain the terminologies in our results as follows; we refer to the books by Kahane [7] and by Falconer [3]. By an N-parameter d-dimensional fractional Brownian motion, "(N,d) fBm" for brevity, it means an N-parameter centered Gaussian process $X(t) = (X_1(t),\dots,X_d(t))$, $t \in \mathbb{R}^N$, with values in $\mathbb{R}^d$, the components $X_j(t)$, $j = 1,\dots,d$, are independent and are distributed as a real-valued centered Gaussian process $Y(t)$ which is with stationary increments and with incremental variance

$$\mathbb{E}[(Y(t)-Y(s))^2] = c_H|t-s|^{2H}, \quad s,t \in R^N.$$

The $H : 0 < H < 1$ is called the Hurst index of the process, and the constant c_H is assumed to be 1. The case $N = 1, H = 1/2$ then determines the Brownian motion in $\mathbb{R}^d$, and the case $N \geq 2, H = 1/2$ is usually referred as a N-parameter Lévy's Brownian motion. We assume, without loss of generality, that the sample paths of an fBm are continuous, and we refer to Chapter 18 of [7] and Chapter 16 of [3] for descriptions of (N,d) fBm's (in [7], his γ there is our $2H$). For any $x \in R^d$ let $\bar{x}$ be the unique point in the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ obtained by taking each component of x mod 1; notice that $\mathbb{T}^1 = [0,1]$. For a process $X(t)$ in $\mathbb{R}^d$, $\bar{X}(t)$ is then the corresponding process in the torus $\mathbb{T}^d$. By a Hausdorff β-set in $\mathbb{R}^m$, it means a compact set $E \subset \mathbb{R}^m$ for which $0 < h_\beta(E) < \infty$; here the $h_\beta(\cdot)$ means Hausdorff β-dimensional measure, and one may see Chapter 10 of [7] and Chapter 2 of [3] for details. The two-fourths Cantor set $C_{4,2}$ in Theorem 1.1 is adapted from p. 132 of [7] (we use his $\xi = 1/4$ there); as it is mentioned there, $h_{1/2}(C_{4,2}) = 1$, so that $C_{4,2}$ is a Hausdorff β-set with $\beta = 1/2$. By an expanding endomorphism T on T^d, we mean a linear transformation $T : \mathbb{T}^d \to \mathbb{T}^d$ such that the singular values of the $d \times d$ matrix A determined by T are strictly greater than 1 (when A is a normal matrix, the expanding means that the absolute value of each eigenvalue of A is strictly greatly than 1).

A perspective: Firstly, we mention that Hochman and Shmerkin [6] has a study on the equidistribution and the normality from fractal measures. Secondly, we remark that, in general, to see the asymptotic behavior of a specific trajectory $T^n x$ in a dynamics is of quiet difficulty, and in this article we illustrate how it may work for *random inputs*. To this end, we would even think (boldly) that, with probability 1, for a.e. $t \in [0,1]$, the sequence $T^n(\bar{B}(t))$, $n = 0,1,2,\dots$, would be an equidistributed sequence, for $\bar{B}(t)$ the Brownian motion living in in the three-dimensional unit spherical surface or a planar fractal (namely the Sierpiński gasket or carpet), and T being a sort of expanding transformations on such structures. In this aspect, we notice Section 6 of [5].

2. Proofs

Proof of Theorem 1.5. We consider the Fourier decay of a chosen measure supported on the image set $X(E) \subset \mathbb{R}^d$. For notational convenience, we work in the context of X, and transfer to $\bar{X}$ is mod 1. Let h_β be the Hausdorff β-dimensional measure in $\mathbb{R}^N$, then we consider the (random) measure μ_ω induced by the h_β and the random mapping X_ω; here we add the random sign ω to emphasize the pathwise definition.

$$\mu_\omega(A) := h_\beta(X_\omega^{-1}(A)), \quad \text{Borel} A \subset \mathbb{R}^d,$$

which is supported on $X_\omega(E)$; it is a finite positive measure in $\mathbb{R}^d$, since we assume that $0 < h_\beta(E) < \infty$. The Fourier transform of μ is

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x}\mu(dx) = \int_{\mathbb{R}^N} e^{2\pi i\xi\cdot X(t)}\, \mathrm{d}h_\beta(t), \quad \xi \in \mathbb{R}^d.$$

Assuming that $\beta \le Hd$, we see that those arguments established on p. 265-7 of Kahane [7] (these arguments lead to the proof of Theorem 1 there, and they have been minor corrected in (2.34) of [11]) assert that, when $\beta \le Hd$, there is the following moment estimate:

$$\mathbb{E}[|\widehat{\mu}(\xi)|^{2p}] \le C^p (h_\beta(E))^p p^{\eta p}\big(2|\xi|^{-1/H}\big)^{\beta p}, \quad \forall \text{ integer } p \ge 1,$$

and the constants C, η depend only on N, H.

This moment estimate then asserts that, with probability 1,

$$\limsup_{|\xi|\to\infty} \frac{|\widehat{\mu}(\xi)|}{\sqrt{(|\xi|^{-2\beta/H})(\ln|\xi|)^{\eta}}} < \infty.$$

Thus, we have the Fourier decay: with probability 1,

$$\widehat{\mu}(\xi) = O(|\xi|^{-2\beta/H}\phi(|\xi|))^{1/2}, \quad |\xi| \to \infty,$$

where $\phi(r)$ is a slowly increasing function in r as $r \uparrow \infty$; refer to Theorem 1 on p. 267 of [7] (distinguish the notations there and here).

Recently, Fraser and Sahlsten [4] extend the Davenport-Erdös-LeVeque Theorem [2] to $\mathbb{T}^d$ for any $d \ge 1$, via the Weyl equidistribution criterion for T^d; see Theorem 1.6 in Section 4 of their paper. We employ this to assert that, for almost sure ω, $T^n x$ ($n = 0, 1, 2, \ldots$) is equidistributed for μ_ω-a.e. x. We then have the assertion of Theorem 1.5, by the definition of our $\mu_\omega := h_\beta(X_\omega^{-1})$.

Proof of Corollary 1.7. We apply Theorem 1.5 to $N = 2, d = 3, H = 1/2$, with $\beta = Hd = 3/2$, and the product $C_{4,2} \times [0, 2\pi]$ is a Hausdorff $(1 + \ln 2/\ln 4 = 3/2)$-set, as it is stated in Chapter 7 of [7] (we use $C_{4,2}$ instead of his middle-thirds Cantor set there). □

We should mention that the arguments in the above fail to hold in case E is a Hausdorff β-set with $\beta > Hd$; indeed, in this case the image of E under the random mapping X, with probability 1, has non-void interior, and the induced measure μ defined above is absolutely continuous w.r.t. Leb_d (see Section 4, Chapter 18 of [7]).

Proof of Theorem 1.1. It is well-known that, for $x \in (0, 1)$, the (simple) normality of x in an integer base $b > 1$ is equivalent to equidistribution of the sequence $x := T^n x \bmod 1$, $n = 0, 1, 2, \ldots$, where $T_b x := b \cdot x \bmod 1$. Thus Theorem 1.1 is directly applicable to get the normality, with $\beta = Hd = 1/2$. One referee indicates to the author that both Theorems 1.1 and 1.4 can be viewed directly from Émile Borel's 1909 Normal Number Theorem via some intriguing arguments, and indeed the arguments would give us a more general statement; we would adapt the indications in the following context, with gratitude. First of all, the mod 1 Gaussian measure and the Lebesgue measure on $[0, 1]$ are mutually absolutely continuous. In the following context, we use the same notation $B(t)$ for the Brownian path and its mod 1 version (that is, we scrape that bar from $\bar{B}(t)$ in statements of Theorems 1.1 and 1.4). From this mutual absolute continuity, for the linear Brownian path $B(t)$, thanks to Borel's Theorem, for each $t > 0$, $B(t)$ is normal in every base with probability 1; notice the order of the quantifiers. Consider the product of the $h_{1/2}$ measure (recall the notations given in Section 1) and the underlying probability measure. By Fubini's Theorem, we have

$$P\{\omega : B(t, \omega) \text{ is normal for } h_{1/2}\text{-a.e. } t \in [0, 1]\} = 1;$$

here and below, the term "normal" means the (simple) normality in every base. Thus, the normality assertion of Theorem 1.1 is obtained, since $C_{4,2}$ is a Hausdorff (1/2)-set, indeed p. 132 of [7] tells us that $h_{1/2}(C_{4,2}) = 1$. The above argument also suggests a more general statement: if $C \subset [0, 1]$ is analytic and is an h_β-set for some $0 < \beta \le 1$ ($\beta = 1$ is the Leb), then the random set $\{t \in C : B(t) \text{ is not normal}\}$ is, with probability 1, h_β-null.

However, to prove assertions of Theorem 1.1 on the size of *Brownian normal numbers* (recall this term in Remark 1.2), we need the crucial role of $C_{4,2}$ as a self-similar set and it critical value $\beta = 1/2$. For the Hausdorff dimension 1 assertion of Theorem 1.1, we use the uniform dimension properties of the real Brownian paths, as shown in Theorem 1.15 of Balka and Peres [1]; we can apply this theorem because the time-set $C_{4,2} \subset [0,1]$ is self-similar by its construction (we notice that the self-similarity is crucial), and $N_{4,2}$ is the (random) subset $C_{4,2}$ with Hausdorff dimension 1/2, since we have shown that, with probability 1, $h_{1/2}(C_{4,2} \backslash N_{4,2}) = 0$; therefore, with probability 1,

$$\dim_H(B(N_{4,2})) = 2\dim_H(N_{4,2}) = 2 \cdot (1/2) = 1.$$

For the Lebesgue measure 0 assertion of Theorem 1.1, we need a precise result for image sets of the linear Brownian motion: for any time-set $A \subset [0,1]$: with probability 1, the image $B(A) \subset \mathbb{R}$ has positive linear Lebesgue measure, if and only if A has positive (1/2)-dimensional Riesz capacity, and the latter implies that $h_{1/2}(A) = \infty$ (the reader may trace those statements and arguments in Chapter 17 (Sections 2 and 3 in particular) of [7] for the proof). We mention that the pioneering article by S. J. Taylor [12] is crucial for the connection of capacity theory to *random* fractals. Now, since $h_{1/2}(C_{4,2}) = 1$, we then have $Leb_1(B(C_{4,2})) = 0$; hence so is $B(N_{4,2})$. □

Proof of Theorem 1.4. For the normality we may apply directly Theorem 1.5, with the endomorphism $(T_b, T_{b'})$ on $\mathbb{T}^2$, for any given two base b, b'; below, we consider again the clever argument indicated by the referee. Let $(B_1(t), B_2(t))$ be the mod 1 planar Brownian motion so that it lives on $\mathbb{T}^2 \simeq [0,1]^2$, consider the random set

$$\mathscr{T}_j = \{t \in [0,1] : B_j(t) \text{ is normal}\},\ j = 1,2.$$

The main content of Theorem 1.4 is to show that

$$\mathscr{R} := \{(B_1(t), B_2(t)) : t \in \mathscr{T}_1 \cap \mathscr{T}_2\}$$

has $\dim_H(\mathscr{R}) = 2$, since the fact that the set has Leb_2 zero is well-known from Paul Lévy's 1937 Zero Area Theorem. By Fubini's argument, as before, we have, with probability 1, $Leb(\mathscr{T}_1 \cap \mathscr{T}_2) = 1$, since each $\mathscr{T}_j$ does (by Borel's Theorem). Thanks to Robert Kaufman's 1969 Uniform Dimension Theorem, we then have $\dim_H(\mathscr{R}) = 2$. We mention that Lévy's Theorem can be seen in Theorems 2.24 and 4.18 of [10], and Kaufman's Theorem can be seen in the beginning of [1]. □

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