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The ergodic theory of geodesic flows over certain manifolds without conjugate points

Weisheng Wu

(Recommended by Boris Hasselblatt)

Abstract. In this article, we announce several new results in the ergodic theory of geodesic flows over uniform visibility manifolds without conjugate points. The topics include the ergodicity with respect to the Liouville measure, uniqueness and Bernoulli properties of the measure of maximal entropy and equilibrium states, counting closed geodesics and volume growth asymptotics, Margulis functions and related rigidity phenomenon, and the Hopf-Tsuji-Sullivan dichotomy with respect to the Bowen-Margulis measure. The detailed proof is given in [37].

1. Introduction

The ergodic theory of geodesic flows is an important topic both in dynamical systems and differential geometry, which has been extensively studied since the beginning of the last century. The geodesic flow on manifolds of negative curvature or nonpositive curvature is the archetype of hyperbolic dynamical systems and chaotic phenomena. Beyond nonpositive curvature, the study of ergodic properties of geodesic flows faces a great challenge and relatively few results are known so far. In this paper, we present new results on ergodic properties of geodesic flows on various classes of manifolds without conjugate points, obtained in [37]. In this case, the geodesic flow only enjoys very weak hyperbolicity as we presently explain.

Let us prepare necessary notations and definitions. Suppose that M is a C^∞ closed n-dimensional manifold equipped with a C^∞ Riemannian metric g, and SM denotes the unit tangent bundle of M. Given $v \in SM$, there exists a unique geodesic $c_v : \mathbb{R} \to M$ satisfying $\dot c_v(0) = v$. The geodesic flow $\phi = (\phi^t)_{t\in\mathbb{R}}$ on SM is defined as

$$\phi^t : SM \to SM, \qquad v \mapsto \dot c_v(t), \quad \forall\ t \in \mathbb{R}.$$

A vector field $J(t)$ along a geodesic $c : \mathbb{R} \to M$ is called a *Jacobi field* if it satisfies the Jacobi equation $J'' + R(J, \dot c)\dot c = 0$, where R is the Riemannian curvature tensor and $'$ denotes the covariant derivative along c. Jacobi fields arise from geodesic variations and represent how nearby geodesics infinitesimally separate.

Definition 1.1. Let c be a geodesic on a Riemannian manifold (M, g).

(1) A pair of distinct points $p = c(t_1)$ and $q = c(t_2)$ is called *focal* if there is a Jacobi field J along c such that $J(t_1) = 0$, $J'(t_1) \neq 0$ and $\frac{d}{dt}\Big|_{t=t_2} \|J(t)\|^2 = 0$;

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(2) $p = c(t_1)$ and $q = c(t_2)$ are called *conjugate* if there is a nontrivial Jacobi field J along c such that $J(t_1) = 0 = J(t_2)$.

The manifold (M, g) is called a manifold *without focal/conjugate points* if there is no focal/conjugate points on any geodesic on M.

It is clear that if a manifold has no focal points then it has no conjugate points. All manifolds of nonpositive curvature always have no focal points. In brief, we have

$$\text{negative curvature} \Rightarrow \text{nonpositive curvature} \Rightarrow \text{no focal points}$$
$$\Rightarrow \text{no conjugate points}.$$

The hyperbolicity of the geodesic flow decreases in this order. The study of geodesic flows in negative curvature was initiated by Artin, Hedlund, Hopf [18, 19], Anosov [1], Margulis [29], etc., and in nonpositive curvature by Pesin [3], Knieper [22], etc.

For geodesic flows on manifolds without conjugate points, the very weak hyperbolicity prevents us from using rich results from hyperbolic dynamics. Many tools in nonpositive curvature, such as convexity, monotonicity, and analysis involving the curvature, are not available in general. Nevertheless, Eberlein [13, 14] introduced the following uniform visibility axiom, which provides certain hyperbolicity at large scales.

Definition 1.2 ([9, Definition 2.1])**.** A simply connected manifold X without conjugate points is a *uniform visibility manifold* if for every $\epsilon > 0$ there exists $L = L(\epsilon) > 0$ such that whenever a geodesic segment $c : [a, b] \to X$ stays at distance at least L from some point $p \in X$, then

$$\angle_p(c) := \sup_{a \le s, t \le b} \angle_p\big(c(s), c(t)\big) < \epsilon.$$

We say that M is a *uniform visibility* manifold if its universal cover is.

Given $v \in SM$, there is a natural isomorphism between the tangent space $T_v(SM)$ and the space of perpendicular Jacobi fields along the geodesic c_v. The *stable/unstable Green bundles* $G^s(v)/G^u(v)$, $v \in SM$ are subbundles of TSM associated to the two distinguished families of Jacobi fields constructed by Green [16], the so called *stable/unstable Jacobi fields*. In general $G^s(v)/G^u(v)$ are only measurable in v, and they are not necessarily transverse to each other at v. We say that M has *continuous Green bundles* or *continuous asymptote* if $G^s(v)/G^u(v)$ are continuous in v; and M has *bounded asymptote* if there exists $C > 0$ such that $\|J^s(t)\| \le C\|J^s(0)\|$ for any $t > 0$ and any stable Jacobi field J^s. We have (cf. [21, 5.3 Satz], [15])

$$\text{no focal points} \Rightarrow \text{bounded asymptote} \Rightarrow \text{continuous asymptote}.$$

We refer to [37] for the definitions of the boundary at infinity ∂X, the Busemann functions and stable/unstable horospherical foliations $\mathscr{F}^s/\mathscr{F}^u$. Both $\mathscr{F}^s(v), \mathscr{F}^u(v)$ are continuous in $v \in SM$. Denote $\mathscr{I}(v) = \mathscr{F}^s(v) \cap \mathscr{F}^u(v)$. If $\mathscr{I}(v) = \{v\}$, then v is called an *expansive vector*. Denote the set of expansive vectors by $\mathscr{R}_0 := \big\{v \in SM : \mathscr{I}(v) = \{v\}\big\}$. Meanwhile, we define

$$\mathscr{R}_1 := \big\{v \in SM : G^s(v) \cap G^u(v) = \{0\}\big\}. \tag{1.1}$$

If M has continuous asymptote, then $\mathscr{R}_1$ is an open subset of SM. The two sets $\mathscr{R}_0$ and $\mathscr{R}_1$ are central objects in our study of dynamical properties of the geodesic flow.

In the following sections, we state the main results in [37] on ergodic properties of the geodesic flow over various classes of manifolds without conjugate points. We refer to [37] for details of the proof and the state of art of the problems.

2. Ergodicity of Liouville measure

The Liouville measure ν is a smooth measure on SM invariant under the geodesic flow. In the 1970s, by using his theory of nonuniform hyperbolicity, Pesin obtained a celebrated result on the ergodicity of ν for the geodesic flow on rank one manifolds without focal points (see e.g. [37] for the definition of rank). Define the sets:

$$\begin{aligned} \Delta^+ &:= \left\{v \in SM : \chi(v,\xi) < 0 \text{ for any } \xi \in G^s(v)\right\}, \\ \Delta^- &:= \left\{v \in SM : \chi(v,\xi) > 0 \text{ for any } \xi \in G^u(v)\right\}, \\ \Delta &:= \Delta^+ \cap \Delta^-, \end{aligned}$$

where $\chi(v,\xi) := \lim_{T\to\infty} \frac{1}{T} \log \|d\phi^t \xi\|$ denotes the Lyapunov exponent. Δ is called the *regular set* with respect to the geodesic flow.

Theorem 2.1 (Pesin, cf. [3])**.** *For the geodesic flow on a smooth, connected and closed surface of genus $\mathfrak{g} \geq 2$ and without focal points, we have that $\nu(\Delta) > 0$, and $\phi^t|_\Delta$ is ergodic.*

However, even for a surface M of genus $\mathfrak{g} \geq 2$ and of nonpositive curvature, the ergodicity of ν on SM is widely open since the 1980s. If a geodesic has zero curvature everywhere, we call it *flat*.

Conjecture 2.2 (cf. [6, 17, 34] etc.)**.** *Let (M, g) be a smooth, connected and closed surface of genus $\mathfrak{g} \geq 2$, which has nonpositive curvature. Then all flat geodesics are closed and there are only finitely many homotopy classes of such geodesics. In particular, $\nu(\Delta^c) = 0$, and hence the geodesic flow on SM is ergodic.*

For early progresses on the conjecture, one can see for example [2, 34]. It is verified for surfaces of genus $\mathfrak{g} \geq 2$ with nonpositive curvature [36] and without focal points [40], assuming that the open subset of M of negative curvature has finitely many connected components.

We consider the ergodicity of the geodesic flows on surfaces of genus $\mathfrak{g} \geq 2$ without conjugate points and with continuous Green bundles. Let $\mathcal{M}_\phi(SM)$ denote the set of all probability measures on SM invariant under the geodesic flow. The following relation between the regular set Δ and the set $\mathcal{R}_1$ in (1.1) in arbitrary dimension is due to Mamani-Ruggiero [28].

Theorem 2.3. *(Cf.* [28, Corollary 7.1]*) Let (M, g) be a smooth, connected and closed visibility manifold without conjugate points and with continuous Green bundles, and $\mu \in \mathcal{M}_\phi(SM)$. Suppose the geodesic flow has a periodic hyperbolic point. Then Δ agrees μ-almost everywhere with the open dense set $\mathcal{R}_1$.*

When M is a closed surface of genus $\mathfrak{g} \geq 2$, by Katok's result [20, Theorem B], the geodesic flow has positive topological entropy, and hence there exists an ergodic measure with positive entropy, which is hyperbolic. Then there exists a hyperbolic periodic point under the geodesic flow (see [20, Theorem 4.1]). This allows us to apply Theorem 2.3.

We call the geodesic c_v *singular* if $v \in \mathcal{R}_1^c$. We must explore the geometric properties of singular geodesics, which are also of independent interest. One essential difficulty is that the flat strip lemma fails in general for surfaces without conjugate points [4], and there is lack of control on the geometric shape of the generalized strips. Nevertheless, we are able to prove that an ideal triangle with certain singular edge has infinite area. This fact is essential for proving that $\mathcal{R}_1^c$ is contained in a finite number of free homotopy classes, given that all singular geodesics are closed.

Our results are summarized in the following theorem. Here $\mathrm{Per}(\phi^t)$ denotes the set of periodic points of the geodesic flow.

Theorem A. *Let (M,g) be a smooth, connected and closed surface of genus $\mathfrak{g} \geq 2$ without conjugate points and with continuous Green bundles, then Δ agrees ν-almost everywhere with the open dense set $\mathscr{R}_1$. If, furthermore, M has bounded asymptote, then the following holds:*

(1) *Let c_v be a lift of a closed geodesic and $v \in \mathscr{R}_1^c$. Then an ideal triangle with c_v as an edge has infinite area;*
(2) *If $\mathscr{R}_1^c \subset Per(\phi^t)$, then there is a finite decomposition*

$$\mathscr{R}_1^c = \mathscr{O}_1 \cup \mathscr{O}_2 \cup \cdots \mathscr{O}_k \cup \mathscr{S}_1 \cup \mathscr{S}_2 \cup \cdots \cup \mathscr{S}_\ell,$$

where each $\mathscr{O}_i, 1 \leq i \leq k$, is an isolated periodic orbit and each $\mathscr{S}_j, 1 \leq j \leq \ell$, consists of vectors tangent to a generalized strip. Here k and ℓ are allowed to be 0. In particular, $\nu(\mathscr{R}_1^c) = 0$, and hence the geodesic flow on SM is ergodic.

3. Patterson-Sullivan construction of Bowen-Margulis measures

The classical variational principle states that the topological entropy of the geodesic flow, denoted by $h_{\text{top}}(\phi^1)$, is the supremum of the metric entropy over all the invariant (or ergodic) probability measures. A measure realizing the supremum is called a *measure of maximal entropy* (MME for short).

Since the geodesic flow is C^∞, by a classical result of Newhouse [30], MME always exists. Climenhaga, Knieper, and War [8] proved the uniqueness of MME for the following manifolds in *class $\mathscr{H}$*, which includes all compact surfaces of genus $\mathfrak{g} \geq 2$ without conjugate points.

Definition 3.1. We say that compact manifolds without conjugate points satisfying the following conditions are in *class $\mathscr{H}$*:

(1) There exists a Riemannian metric g_0 on M for which all sectional curvatures are negative;
(2) M has uniform visibility property;
(3) The fundamental group $\pi_1(M)$ is residually finite: the intersection of its finite index subgroups is trivial;
(4) There exists $h_0 < h_{\text{top}}(\phi^1)$ such that any ergodic probability measure μ on SM with entropy strictly greater than h_0 is almost expansive (cf. [9, Definition 2.8]).

Climenhaga, Knieper, and War [8] used the Morse Lemma to get coarse estimates, and successfully applied Climenhaga-Thompson's criterion [10] using a nonuniform version of specification and expansiveness, to obtain the uniqueness of the MME.

Condition (1) in the definition of class $\mathscr{H}$ imposes a rather restrictive condition on the topology of the manifold in higher dimension. Mamani-Ruggiero [28] have removed this condition:

Theorem 3.2 ([28, Theorem 1.2])**.** *Let (M,g) be a closed C^∞ uniform visibility manifold without conjugate points. Suppose that*

$$\sup\{h_\mu(\phi^1) : \mu \textit{ is supported on } \mathscr{R}_0^c\} < h_{\text{top}}(\phi^1).$$

Then the geodesic flow has a unique MME.

The entropy gap assumption above is equivalent to the condition (4) in the definition of class $\mathscr{H}$. Then Theorem 3.2 applies to a class of manifolds larger than class $\mathscr{H}$, since conditions (1) and (3) in the definition of class $\mathscr{H}$ are dropped. Mamani and Ruggiero [28] proved that there exists an expansive factor of the geodesic flow acting on a compact metric space, which is topologically mixing with local product structure and hence has unique MME.

3.1. Patterson-Sullivan measure. For rank one manifolds of nonpositive curvature, Knieper [22] provided a Patterson-Sullivan construction of the unique MME. Climenhaga, Knieper, and War [8] constructed a MME via Patterson-Sullivan measures under conditions (1) and (2) in the definition of class $\mathscr{H}$. Clotet [11] also constructed Patterson-Sullivan measures for quasi-convex manifolds satisfying that geodesic rays diverge.

We first give a Patterson-Sullivan construction of MME under condition (2). Under additional condition (4), this MME will be the unique MME in Theorem 3.2. Patterson-Sullivan construction of MME is the starting point for almost all the subsequent results, e.g., Theorems B, D, E, I.

Definition 3.3. Let X be a simply connected uniform visibility manifold without conjugate points and Γ a discrete subgroup of $\mathrm{Iso}(X)$, the group of isometries of X. For a given constant $r>0$, a family of finite Borel measures $\{\mu_p\}_{p\in X}$ on ∂X is called an r-dimensional *Busemann density* if

(1) for any $p,\ q\in X$ and μ_p-a.e. $\xi\in\partial X$,
$$\frac{d\mu_q}{d\mu_p}(\xi)=e^{-r\cdot b_\xi(q,p)}$$
where $b_\xi(q,p)$ is the Busemann function;

(2) $\{\mu_p\}_{p\in X}$ is Γ-equivariant, i.e., for all Borel sets $A\subset\partial X$ and for any $\gamma\in\Gamma$, we have $\mu_{\gamma p}(\gamma A)=\mu_p(A)$.

In [27], the authors constructed Busemann density via Patterson-Sullivan construction. For each pair of points $(p,p_0)\in X\times X$ and $s\in\mathbb{R}$, *Poincaré series* is defined as
$$P(s,p,p_0):=\sum_{\alpha\in\Gamma}e^{-s\cdot d(p,\alpha p_0)}.$$
The number $\delta_\Gamma:=\inf\{s\in\mathbb{R}\mid P(s,p,p_0)<+\infty\}$ is called the *critical exponent* of Γ. By the triangle inequality, it is easy to check that δ_Γ is independent of p and p_0. We say Γ is of *divergent type* if the Poincaré series diverges when $s=\delta_\Gamma$.

Fix a point $p_0\in X$. For each $s>\delta_\Gamma$ and $p\in X$, consider the measure
$$\mu_{p,s}:=\frac{\sum_{\alpha\in\Gamma}\delta_{\alpha p_0}e^{-s\cdot d(p,\alpha p_0)}}{\sum_{\alpha\in\Gamma}e^{-s\cdot d(p_0,\alpha p_0)}},$$
where $\delta_{\alpha p_0}$ is the Dirac measure at point αp_0. The following properties of $\mu_{p,s}$ can be easily checked by using the triangle inequality:

(1) $e^{-s\cdot d(p,p_0)}\le\mu_{p,s}(\overline{X})\le e^{s\cdot d(p,p_0)}$.
(2) $\Gamma p_0\subset\mathrm{supp}(\mu_{p,s})\subset\overline{\Gamma p_0}$.

One may assume that Γ is of divergent type. When Γ is of convergent type, Patterson showed in [32] that one can weight the Poincaré series with a positive monotone increasing function to make the refined Poincaré series diverge at δ_Γ.

Now for $p\in X$, consider a weak* limit
$$\lim_{s_k\downarrow\delta_\Gamma}\mu_{p,s_k}=\mu_p.$$
Since $P(s,p_0,p_0)$ is divergent for $s=\delta_\Gamma$ and Γ is discrete, $\mathrm{supp}(\mu_p)\subset\overline{\Gamma p_0}\cap\partial X$.

Proposition 3.4. *There is a sequence $s_k\to\delta_\Gamma$ as $k\to\infty$ such that for every $p\in X$ the weak* limit $\lim_{k\to\infty}\mu_{p,s_k}=:\mu_p$ exists. Moreover, $\{\mu_p\}_{p\in X}$ is a δ_Γ-dimensional Busemann density.*

The proof is identical to that of [8, Proposition 5.1]. We do not need to assume that X/Γ is compact.

3.2. Bowen-Margulis measure. Let $P : SX \to \partial X \times \partial X$ be the projection given by $P(v) = (v^-, v^+)$, where v^- (resp. v^+) denotes the asymptotic class of geodesic ray c_v (resp. c_{-v}) in ∂X. Denote by $\partial^2 X := P(SX)$ the subset of pairs in $\partial X \times \partial X$ which can be connected by a geodesic. Note that the connecting geodesic may not be unique. Since X has uniform visibility, every pair $\xi \neq \eta$ in ∂X can be connected by a geodesic, i.e.,

$$\partial^2 X = (\partial X \times \partial X) \setminus \{(\xi,\xi) : \xi \in \partial X\}.$$

Fix a point $p \in X$, we can define a Γ-invariant measure $\overline{\mu}$ on $\partial^2 X$ by

$$d\overline{\mu}(\xi,\eta) := e^{\delta_\Gamma \cdot \beta_p(\xi,\eta)} d\mu_p(\xi) d\mu_p(\eta),$$

where $\beta_p(\xi,\eta) := -(b_\xi(q,p) + b_\eta(q,p))$ is the *Gromov product*, and q is any point on a geodesic c connecting ξ and η.

Let $M = X/\Gamma$. Define an equivalence relation on SM by writing $v \sim w$ iff $w \in \mathscr{I}(v)$. Write $[v] = \mathscr{I}(v)$ for the equivalence class of v. Let $\tilde{\chi} : SX \to SX/\sim$ and $\chi : SM \to SM/\sim$ be the quotient maps. These are continuous maps when we equip the quotient spaces with the metric

$$d([v],[w]) = \min\{d(v',w') : v' \in [v], w' \in [w]\}.$$

The geodesic flow ϕ^t takes equivalence classes to equivalence classes, i.e., $\phi^t[v] = [\phi^t v]$. Hence it induces a flow $\psi^t : SM/\sim \to SM/\sim$.

Theorem 3.5 ([28, Theorem 1.1]). *Let M be a compact C^∞ n-dimensional manifold without conjugate points and with uniform visibility universal cover X. Then the geodesic flow ϕ^t is time-preserving semi-conjugate to the continuous flow ψ^t acting on a compact metric space $SM/\sim$ of topological dimension at least n. Moreover, ψ^t is expansive, topologically mixing and has a local product structure.*

We do not need compactness of M in the following construction. Since $\mathscr{I}(v)$ is compact and SM is a Polish space, the measurable selection theorem of Kuratowski and Ryll-Nardzewski [35, Section 5.2] guarantees the existence of a Borel measurable map $V : SM/\sim \to SM$ such that $\chi \circ V$ is the identity. Recall that $\mathrm{pr} : SX \to SM$ is the natural projection. Then for every $v \in SX$, $\mathrm{pr}^{-1}(V(\mathrm{pr}[v]))$ intersects $[v]$ in a single point, which we denote by $\tilde{V}([v])$. We conclude that $\tilde{V} : SX/\sim \to SX$ is a measurable map such that $\tilde{\chi} \circ \tilde{V}$ is the identity, and moreover $\tilde{V}(\gamma_*[v]) = \gamma_* \tilde{V}([v])$ for any $\gamma \in \Gamma$. Now define a measure $\lambda_{\xi,\eta}$ on each $P^{-1}(\xi,\eta)$ by fixing any $v \in P^{-1}(\xi,\eta)$ and putting, for a Borel measurable set $A \subset SX$,

$$\lambda_{\xi,\eta}(A) := \mathrm{Leb}\{t \in \mathbb{R} : \tilde{V}([\phi^t v]) \in A\},$$

which is independent of the choice of v. Then define a measure M on SX by

$$m(A) = \int_{\partial^2 X} \lambda_{\xi,\eta}(A) d\overline{\mu}(\xi,\eta).$$

Observe that M is Γ-invariant, and it descends to a Borel measure m_Γ on SM.

The measure m_Γ is not necessarily invariant under the geodesic flow, since along a flow orbit inside a generalized strip, the choice for the value of measurable function V may vary. However, the measure $\underline{m_\Gamma} = \chi_* m_\Gamma$ is a flow invariant measure on $SM/\sim$ because $\chi_* \lambda_{\xi,\eta}$ is flow invariant on each $P^{-1}(\xi,\eta)/\sim$.

Now let us suppose that M is compact and then clearly m_Γ is finite. Without loss of generality we scale the measure so that $m_\Gamma(SM) = 1$. Then the set $\chi_*^{-1}\underline{m_\Gamma}$ of Borel probability measures is weak* compact and closed under $(\phi^t)_*$ for all $t \in \mathbb{R}$. So the usual argument from the Krylov-Bogolyubov theorem shows that there is a flow invariant Borel probability measure μ on SM with $\chi_*\mu = \underline{m}_\Gamma$. This lifts to a Γ-invariant and flow invariant Borel measure $\tilde{\mu}$ on SX by

$$\tilde{\mu}(A) = \int_{SM} \#(\mathrm{pr}^{-1}(\underline{v}) \cap A)\, d\mu(\underline{v}).$$

We will call both m_Γ and μ *Bowen-Margulis measures* of the geodesic flow on SM.

Assume that $M = X/\Gamma$ is a compact uniform visibility manifold with universal cover X. Then we can show that $\delta_\Gamma = h_{\mathrm{top}}(\phi^1)$, and the Bowen-Margulis measure μ is a MME. So it coincides with the unique MME under the setting of Theorem 3.2.

3.3. Bernoullicity of MME. The above Patterson-Sullivan construction gives a local product structure of the Bowen-Margulis measure. As an application, using the Ornstein-Weiss argument [31], we can improve the Kolmogorov property of the Bowen-Margulis measure to the Bernoulli property.

Theorem B. *Let (M, g) be a closed C^∞ uniform visibility manifold without conjugate points. Suppose that*

$$\sup\left\{h_\mu(\phi^1) : \mu \text{ is supported on } \mathscr{R}_0^c\right\} < h_{\mathrm{top}}(\phi^1).$$

Then the unique MME is fully supported and has the Bernoulli property.

4. Uniqueness of equilibrium states

Recently, Burns, Climenhaga, Fisher and Thompson [5] proved the uniqueness of equilibrium states for certain potentials in nonpositive curvature, via Climenhaga-Thompson approach [10] using nonuniform specification and expansiveness. Lima-Poletti [23] reproved the uniqueness of equilibrium states by developing a *countable* symbolic coding of the systems as follows. See [23, Sections 2.1 and 2.2] for the definitions and notations of topological Markov flow and homoclinic relation of measures.

Theorem 4.1. *(*[23, Main Theorem]*) Let $\varphi^t : N \to N$ be a flow generated by a $C^{1+\beta}$-vector field Z with $Z \neq 0$ everywhere, and $\lambda > 0$. If μ_1, μ_2 are homoclinically related λ-hyperbolic ergodic measures, then there is an irreducible countable topological Markov flow (Σ_r, σ_r) and a Hölder continuous map $\pi_r : \Sigma_r \to N$ such that:*

1. *$r : \Sigma \to \mathbb{R}^+$ is Hölder continuous and bounded away from zero and infinity.*
2. *$\pi_r \circ \sigma_r^t = \varphi^t \circ \pi_r$ for all $t \in \mathbb{R}$.*
3. *$\pi_r(\Sigma_r^\#)$ has full measure with respect to μ_1 and μ_2.*
4. *Every $x \in N$ has finitely many preimages in $\Sigma_r^\#$.*

To the best of our knowledge, the following is the first result on the uniqueness of equilibrium states for manifolds without conjugate points.

Theorem C. *Let (M, g) be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles, and let $\psi : SM \to \mathbb{R}$ be Hölder continuous or of the form $\psi = q\psi^u$ for $q \in \mathbb{R}$. Suppose that the geodesic flow has a hyperbolic periodic point. If $P(\mathscr{R}_1^c, \psi) < P(\psi)$, then ψ has a unique equilibrium state μ, which is hyperbolic, fully supported and has the Bernoulli property.*

In the above theorem, ψ^u is the *geometric potential* defined as

$$\psi^u(v) := -\frac{d}{dt}\bigg|_{t=0} \log\det\left(d\phi^t|_{G^u(v)}\right).$$

By the minimality of the foliations $\mathscr{F}^s/\mathscr{F}^u$, one can show that any two hyperbolic equilibrium states μ_1, μ_2 are homoclinically related. It turns out that we are able to apply Theorem 4.1 to prove Theorem C.

The assumption of continuity of Green bundles and the existence of a hyperbolic periodic point first appear in [28, Theorem 1.4]. All rank one manifolds of nonpositive curvature satisfy this assumption, though they do not necessarily have uniform visibility. Under this assumption, it is proved in [28, Lemma 10.4] that the entropy gap condition $h_{\text{top}}(\phi^1, \mathscr{R}_1^c) < h_{\text{top}}(\phi^1)$ is fulfilled. Indeed, the quotient flow $(SM/\sim, \psi^t)$ has a unique MME with full support. Since any ergodic MME of the geodesic flow projects to the MME of ψ^t, the former also has full support, and therefore cannot be supported on $\mathscr{R}_1^c$. Then applying the abstract criterion in [7], Mamani and Ruggiero proved the uniqueness of MME of the geodesic flow in this setting [28, Theorem 1.4]. Instead we can apply Theorem C and recover [28, Part 2 of Theorem 1.4].

Theorem 4.2. *Let (M, g) be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles. Suppose that the geodesic flow has a hyperbolic periodic point. Then the geodesic flow has a unique MME, which is hyperbolic, fully supported and has the Bernoulli property.*

In the setting of Theorem 4.2, we will explore some counting and rigidity results in Theorems D, E, F, G, H.

5. Counting closed geodesics and volume asymptotics

5.1. Counting closed geodesics. Let $P(t)$ be the set of free-homotopy classes containing a closed geodesic with length at most $t > 0$. If M has negative curvature, each free-homotopy class contains exactly one closed geodesic, so $\#P(t)$ is just the number of closed geodesics with length at most t. If M has nonpositive curvature or without focal/conjugate points, a free-homotopy class may have infinitely many closed geodesics.

We obtain the following Margulis-type asymptotic estimates of $\#P(t)$.

Theorem D. *Let (M, g) be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles. Suppose that the geodesic flow has a hyperbolic periodic point. Then*

$$\#P(t) \sim \frac{e^{ht}}{ht} \tag{5.1}$$

where $h = h_{top}(\phi^1)$ denotes the topological entropy of the geodesic flow, and the notation $f_1 \sim f_2$ means that $\lim_{t\to\infty} f_1(t)/f_2(t) = 1$.

See [22, 33, 9, 39] for recent developments on this topic beyond negative curvature. Climenhaga, Knieper, and War [9] obtained (5.1) for manifolds without conjugate points in class $\mathscr{H}$. We have dropped the rather restrictive condition (1) as well as condition (3) in the definition of class $\mathscr{H}$, but with an extra assumption of continuous asymptote.

The following is a result of equidistribution of periodic orbits with respect to the Bowen-Margulis measure, which is crucial to prove Theorem D.

Theorem 5.1. *Let (M, g) be as in Theorem D, and $\epsilon \in (0, inj(M)/2)$ is fixed where $inj(M)$ is the injectivity radius of M. For $t > 0$, let $C(t)$ be any maximal set of pairwise non-free-homotopic closed geodesics with lengths in $(t-\epsilon, t]$, and define the measure*

$$\nu_t := \frac{1}{\#C(t)} \sum_{c\in C(t)} \frac{Leb_c}{t}$$

where Leb_c is the Lebesgue measure along the curve $\dot c$ in the unit tangent bundle SM.

Then the measures ν_t converge in the weak* topology to the unique MME as $t \to \infty$.

To prove Theorem D, we follow Margulis' approach and combine ideas in [33, 9]. We construct a flow box in a geometric way which helps us to apply the local product structure of the Bowen-Margulis measure. As a consequence of Theorem 5.1, it is sufficient to count closed geodesics through a fixed geometric flow box. The assumption that the visibility manifold M has continuous Green bundles allows us to choose the flow box inside $\mathscr{R}_1$ to avoid generalized strips. Then we use mixing properties of the MME to count the number of self-intersections of the box under the geodesic flow. To produce closed geodesics from such self-intersections, a type of closing lemma is of particular importance, which is a nontrivial consequence of the uniform visibility property of M.

5.2. Volume growth asymptotics. We also study the twin problem, that is, volume growth asymptotics for uniform visibility manifolds without conjugate points and with continuous Green bundles. See [25, 38] for recent results on this problem.

Theorem E. *Let (M, g) be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles, and X its universal cover. Suppose that the geodesic flow has a hyperbolic periodic point. Then*

$$b_t(x)/\frac{e^{ht}}{h} \sim c(x),$$

where $b_t(x)$ is the Riemannian volume of the ball of radius $t > 0$ around $x \in X$, and $c : X \to \mathbb{R}$ is a continuous function.

Our proof of Theorem E also uses Margulis' approach. First we establish the asymptotics formula for a pair of flow boxes using the mixing property of the MME and the scaling property of the Patterson-Sullivan measure. But then we need to consider countably many pairs of flow boxes and deal with an issue of nonuniformity. We will use the fact that the unique MME is supported on $\mathscr{R}_1^c$ and apply Knieper's techniques. The assumption of continuous Green bundles is crucial here.

6. Margulis function and rigidity

The continuous function $c(x)$ obtained in Theorem E is called *Margulis function.* It is Γ-invariant and hence descends to a function on M, still denoted by c. As Margulis function is closely related to Patterson-Sullivan measures, its interactions with Bowen-Margulis measure, Lebesgue measure and harmonic measures usually give various characterizations of locally symmetric spaces.

6.1. Smoothness of Margulis function. In negative curvature, Katok conjectured that $c(x)$ is almost always not constant and not smooth, unless M is locally symmetric. In [41] Yue answered Katok's conjecture, and his results can be extended to manifolds without conjugate points.

Theorem F. *Let M be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles, X the universal cover of M. Suppose that the geodesic flow has a hyperbolic periodic point. Then*

(1) the Margulis function c is a C^1 function;

(2) if $c(x) \equiv C$, then for any $x \in X$,

$$h = \int_{\partial X} tr U(x,\xi)\, d\tilde{\mu}_x(\xi)$$

where $U(x,\xi)$ and $tr U(x,\xi)$ are the second fundamental form and the mean curvature of the horosphere $H_x(\xi)$ respectively, and $\tilde{\mu}_x$ is the normalized Patterson-Sullivan measure.

We note that the assumption of continuous asymptote in Theorem F guarantees that the Busemann function is C^2.

6.2. Rigidity in dimension two. When $c(x) \equiv C$, we expect stronger rigidity results to hold. To achieve this goal, Yue [41] studied the uniqueness of harmonic measures associated to the strong stable foliation of the geodesic flow in negative curvature. However, as far as we know, the uniqueness of harmonic measures in nonpositive curvature is unknown in higher dimension.

Nevertheless, if $\dim M = 2$, any ball $B^s(x, r)$ of radius r in a strong stable manifold is just a curve, and hence $\mathrm{Vol}(B^s(x, r)) = 2r$. Thus the leaves of the strong stable foliation have polynomial growth. Recently, Clotet [12] proved the unique ergodicity of the horocycle flow on surfaces of genus $\mathfrak{g} \geq 2$, without conjugate points and with continuous Green bundles. Combining with this result, we can show that there is a unique harmonic measure associated to the strong stable foliation. Then we obtain the following theorem by applying Katok's entropy rigidity result [20, Theorem B].

Theorem G. *Let M be a closed Riemannian surface of genus $\mathfrak{g} \geq 2$ without conjugate points and with continuous Green bundles. Then $c(x) \equiv C$ if and only if M has constant negative curvature.*

Remark 6.1. In fact, it is a well known conjecture [41, p.179] in negative curvature that $c(x) \equiv C, x \in M$ if and only if M is locally symmetric. The conjecture is true for surfaces of negative curvature by [41, Theorem 4.3]. Theorem G verifies the conjecture for a more general class of surfaces without conjugate points.

6.3. Flip invariance and rigidity. In arbitrary dimensions, we can still obtain some rigidity results without the uniqueness of harmonic measures. The flip map $F : SM \to SM$ is defined as $F(v) := -v$. For general uniform visibility manifolds, the Bowen-Margulis measure m_Γ is constructed by using measurable choice theorem, hence is not necessarily flip invariant. However, for uniform visibility manifolds with continuous Green bundles and the geodesic flow admitting a hyperbolic periodic point, the Bowen-Margulis measure m_Γ is supported on $\mathscr{R}_1$ and hence it is indeed flip invariant. In this case, consider the conditional measures $\{\bar{\mu}_x\}_{x\in M}$ of m_Γ with respect to the partition $SM = \bigcup_{x\in M} S_x M$. $\bar{\mu}_x$ can be identified as measures on ∂X, and it would be natural to consider whether $\bar{\mu}_x$ and the normalized Patterson-Sullivan measures $\tilde{\mu}_x$ coincide. Yue [42] obtained related rigidity results in negative curvature, which can be generalized as follows.

Theorem H. *Let M be a closed C^∞ uniform visibility manifold without conjugate points and with continuous Green bundles. Suppose that the geodesic flow admits a hyperbolic periodic point. Then the conditional measures $\{\bar{\mu}_x\}_{x\in M}$ of the Bowen-Margulis measure coincide almost everywhere with the normalized Patterson-Sullivan measures $\tilde{\mu}_x$ if and only if M is locally symmetric.*

The assumption of continuous asymptote above again guarantees that the Busemann function is C^2. Both Theorem G and Theorem H give characterizations of rank one locally symmetric spaces among certain closed manifolds without conjugate points.

7. The Hopf-Tsuji-Sullivan dichotomy

We establish the Hopf-Tsuji-Sullivan dichotomy with respect to the Bowen-Margulis measure for the geodesic flow over certain manifolds without conjugate points. First, let us recall some necessary terminology (see [24]).

Let Ω be a locally compact and σ-compact Hausdorff topological space and $\{\phi^t\}_{t\in\mathbb{R}}$ a flow on Ω. A point $\omega \in \Omega$ is said to be *positively/negatively recurrent* if there exists a sequence $t_n \to +\infty$ such that $\phi^{\pm t_n}\omega \to \omega$. On the other hand, $\omega \in \Omega$ is said to be *positively/negatively divergent* if for every compact set $K \subset \Omega$ there exists $T > 0$ such that $\phi^{\pm t}\omega \notin K$ for all $t \geq T$.

Assume that μ is a Borel measure on Ω invariant by the flow $\{\phi^t\}_{t\in\mathbb{R}}$. Then the Hopf decomposition theorem (see [18, Satz 13.1]) says that Ω decomposes into a disjoint union of two ϕ^t-invariant Borel sets Ω_C and Ω_D satisfying the following properties:

(1) There does not exist a Borel subset $E \subset \Omega_C$ with $\mu(E) > 0$ and such that the sets $\{\phi^k E\}_{k\in\mathbb{Z}}$ are pairwise disjoint.
(2) There exists a Borel set $W \subset \Omega_D$ such that Ω_D is the disjoint union of sets $\{W_k\}_{k\in\mathbb{Z}}$, where each W_k is a translate of W under the flow $\{\phi^t\}_{t\in\mathbb{R}}$.

According to Poincaré's recurrence theorem (see [18, Satz 13.2]), μ-almost every point of Ω_C is positively and negatively recurrent. On the other hand, by Hopf's divergence theorem (see also [18, Satz 13.2]), μ-almost every point of Ω_D is positively and negatively divergent. This implies in particular that the sets Ω_C and Ω_D are unique up to sets of measure zero.

The dynamical system $(\Omega, \{\phi^t\}_{t\in\mathbb{R}}, \mu)$ is said to be *completely conservative* if $\mu(\Omega_D) = 0$, and *completely dissipative* if $\mu(\Omega_C) = 0$.

Finally, a dynamical system $(\Omega, \{\phi^t\}_{t\in\mathbb{R}}, \mu)$ is called *ergodic* if every ϕ^t-invariant Borel set $E \subset \Omega$ satisfies either $\mu(E) = 0$ or $\mu(\Omega \setminus E) = 0$. Hence if $(\Omega, \{\phi^t\}_{t\in\mathbb{R}}, \mu)$ is ergodic, then it is either completely conservative or completely dissipative. The latter can only occur for an infinite measure μ which is supported on a single orbit $\{\phi^t\omega : t \in \mathbb{R}\}$ with $\omega \in \Omega$.

Our main result is the following Hopf-Tsuji-Sullivan dichotomy.

Theorem I. *Let X be a simply connected smooth uniform visibility manifold without conjugate points. Suppose that $\Gamma \subseteq Is(X)$ is a non-elementary discrete group which contains an expansive isometry. Let m_Γ be a Bowen-Margulis measure on $SM = SX/\Gamma$ and μ_o the Patterson-Sullivan measure normalized so that $\mu_o(\partial X) = 1$. Then exactly one of the following two complementary cases holds, and the statements (i) to (iii) are equivalent in each case. If furthermore X has bounded asymptote and Γ contains a regular isometry, then the statements (i) to (iv) are equivalent in each case.*

(1) Cases:
- *(i) $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(o,\gamma o)}$ converges.*
- *(ii) $\mu_o(L_\Gamma^{rad}) = 0$, where L_Γ^{rad} denotes the radial limit set of Γ.*
- *(iii) $(SM, (\phi^t)_{t\in\mathbb{R}}, m_\Gamma)$ is completely dissipative.*
- *(iv) $(SM, (\phi^t)_{t\in\mathbb{R}}, m_\Gamma)$ is not ergodic.*

(2) Cases:
- *(i) $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(o,\gamma o)}$ diverges.*
- *(ii) $\mu_o(L_\Gamma^{rad}) = 1$.*
- *(iii) $(SM, (\phi^t)_{t\in\mathbb{R}}, m_\Gamma)$ is completely conservative.*
- *(iv) $(SM, (\phi^t)_{t\in\mathbb{R}}, m_\Gamma)$ is ergodic.*

Remark 7.1. See [26, 24, 27] for recent results beyond negative curvature. We do not require that X is regular as in [27], i.e., for any two points on ∂X, there is at most one geodesic connecting them.

Remark 7.2. Our Bowen-Margulis measure is constructed as a product of Patterson-Sullivan measures and a measure $\lambda_{\xi,\eta}$ supported on a generalized strip $(\xi\eta)$, where a measurable choice theorem is used. Unfortunately, for noncompact manifolds, argument of Krylov-Bogolyubov type cannot be applied, and such Bowen-Margulis measure m_Γ is not necessarily invariant under the geodesic flow. Nevertheless, the projection of m_Γ to the quotient space is invariant under the quotient geodesic flow. So strictly speaking, we are talking about complete dissipativity/conservativity and ergodicity for the quotient geodesic flow $([SM],\{[\phi^t]\}_{t\in\mathbb{R}},[m_\Gamma])$ in Theorem I. Since there is no confusion, we still state our results for the geodesic flow $(SM,\{\phi^t\}_{t\in\mathbb{R}},m_\Gamma)$.

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