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Equivariant asymptotics on Grauert tubes

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(Recommended by Tobias Colding)

ABSTRACT. We report on recent work on the scaling asymptotics of the equivariant components of Poisson and Szegö kernels on the Grauert tube boundaries associated to a real-analytic Riemannian manifold acted upon by a compact Lie group. Building largely on techniques of Zelditch and Chang and Rabinowitz, we describe the asymptotic concentration along the zero locus of the moment map of the equivariant eigenfunctions of a Toeplitz operator associated to the homogeneous geodesic flow and of the complexified equivariant eigenfunctions of the Laplacian. We also digress on some applications.

1. Introduction

A d-dimensional compact real-analytic manifold M admits an essentially unique complexification, that is, a complex manifold $(\widetilde{M}, J)$ (here J is the complex structure) of complex dimension d, in which M sits as a totally real submanifold [3]; one calls $\widetilde{M}$ the *Bruhat–Whitney complexification* of M. Furthermore, the choice of a real-analytic Riemannian metric κ on M determines a unique real-analytic function $\rho : \widetilde{M} \to \mathbb{R}$ with the following properties ([13, 14, 17, 18, 31, 24, 25]):

(1) $\rho \geq 0$ and $\rho^{-1}(0) = M$;
(2) ρ is strictly plurisubharmonic, from which $\Omega := \imath\,\partial\overline{\partial}\rho$ is a Kähler form, inducing a Riemannian metric $\hat{\kappa} := \Omega(\cdot, J\cdot)$ on $\widetilde{M}$;
(3) (M, κ) is a Riemannian submanifold of $(\widetilde{M}, \widehat{\kappa})$;
(4) $\sqrt{\rho}$ satisfies the complex homogeneous Monge–Ampère equation on $\tilde{M} \setminus M$:

$$\det\left(\frac{\partial^2 \sqrt{\rho}}{\partial z_k \bar{\partial} z_j}\right) = 0.$$

An alternative perspective is to say that κ determines a unique complex structure J_{ad} (called *adapted* in [18]) on a neighbourhood of the zero section of the tangent bundle, with the property that the parametrized leaves of the Riemannian foliation are holomorphic. We shall identify the tangent and cotangent bundles TM and $T^{\vee}M$ of M by means of κ.

As shown in [18] and [16], the equivalence between these approaches is realized by the so-called imaginary time exponential map of (M, κ), which maps a tubular neighbourhood of the zero section in TM to a tubular neighbourhood of M in $\widetilde{M}$, and is the identity on M; this map is (J_{ad}, J)-holomorphic, and intertwines ρ with the square norm function $\|\cdot\|^2$. Furthermore, it is a symplectomorphism with respect to the canonical

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symplectic structure Ω_{can} on the (co)tangent bundle and Ω on $\widetilde{M}$. In short, for any $\tau > 0$ sufficiently small if $T^\tau M \subset TM$ is the locus where the norm is $< \tau$ and $\widetilde{M}^\tau := \rho^{-1}\left(0, \tau^2\right) \subseteq \widetilde{M}$, then the imaginary time exponential map is an isomorphism between the Kähler manifolds $\left(T^\tau M, J_{\mathrm{ad}}, \Omega_{can}\right)$ and $\left(\widetilde{M}^\tau, J, \Omega\right)$.

We shall set $\tau_{\max} := \sup \rho$; then $0 < \tau_{\max} \le +\infty$, and $\tau_{\max} < +\infty$ in the presence of negative sectional curvatures [18].

For any $\tau \in (0, \tau_{\max}]$, one calls $\widetilde{M}^\tau$ the *Grauert tube* of radius τ. If $\tau \in (0, \tau_{\max})$, by construction $\widetilde{M}^\tau$ is a strictly pseudoconvex domain in $\widetilde{M}$. On its boundary $X^\tau := \partial \widetilde{M}^\tau$ we have the following objects:

(1) a contact form α^τ, given by the restriction of $\alpha := \Im(\partial\rho)$;
(2) the Hardy space $H(X^\tau) \subseteq L^2(X^\tau)$ (we adopt the choice of volume form on X^τ in [23, §3.2.3]);
(3) the Szegö projector $\Pi^\tau : L^2(X^\tau) \to H(X^\tau)$, and its distributional kernel $\Pi^\tau(\cdot,\cdot) \in \mathscr{D}'(X^\tau \times X^\tau)$ (everything that follows is based on the description in [2] of Π^τ as a Fourier integral operator with a complex phase ψ^τ of positive type);
(4) the Hamiltonian vector field $\upsilon_{\sqrt{\rho}}$ of $\sqrt{\rho}$ with respect to Ω, and its restriction $\upsilon^\tau_{\sqrt{\rho}}$ to X^τ;
(5) the formally self-adjoint differential operator $D^\tau_{\sqrt{\rho}} := \imath\, \upsilon_{\sqrt{\rho}}$;
(6) the Toeplitz operator

$$\mathfrak{D}^\tau_{\sqrt{\rho}} := \Pi^\tau \circ D^\tau_{\sqrt{\rho}} \circ \Pi^\tau. \tag{1.1}$$

Remark 1.1. The flow $\Gamma^\tau_t : X^\tau \to X^\tau$ of $\upsilon_{\sqrt{\rho}}$ is intertwined by the imaginary time exponential map with the homogeneous geodesic flow on the tangent bundle; for this reason, we shall somewhat abusively refer to it as 'the geodesic flow on X^τ'.

The spectral analysis of $\mathfrak{D}^\tau_{\sqrt{\rho}}$ on $H(X^\tau)$ was pioneered by Zelditch [34, 38] and may be viewed heuristically as a counterpart in the complex domain of the spectral analysis of the Laplacian on M; compression with Π^τ is necessary, since the flow of $\upsilon^\tau_{\sqrt{\rho}}$ is generally not CR-holomorphic.

Consider the distinct eigenvalues $\lambda^\tau_1 < \lambda^\tau_2 < \cdots \uparrow +\infty$ of $\mathfrak{D}^\tau_{\sqrt{\rho}}$; for every $j = 1, 2, \ldots$ let $\Pi^\tau_j : L^2(X^\tau) \to V^\tau_j$ be the orthogonal projector onto the eigenspace $V^\tau_j \subseteq H(X^\tau)$ of λ^τ_j. Since V^τ_j is finite-dimensional, Π^τ_j is smoothing, i.e., the distributional kernel $\Pi^\tau_j(\cdot,\cdot) \in \mathscr{C}^\infty(X^\tau \times X^\tau)$. In the following, we shall simplify notation and write $\lambda_j = \lambda^\tau_j$, $V_j = V^\tau_j$, thus leaving dependence on τ of spectral data implicit. The asymptotic distribution of the λ_j's and the local concentration of the corresponding eigenfunctions is probed by smoothing operator kernels of the form

$$\Pi^\tau_{\chi,\lambda} := \sum_j \widehat{\chi}(\lambda - \lambda_j)\, \Pi^\tau_j, \tag{1.2}$$

where $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ is of compact support and $\lambda \to +\infty$.

Smoothed spectral projectors as (1.2) (and its equivariant version (1.4) below) are commonly studied in spectral theory (see, e.g., [11, 6] for pseudodifferential operators; in the setting of Toeplitz operators, with an emphasis on local scaling asymptotics, see, e.g., [19, 20, 21, 22, 34, 38, 39, 40, 41]).

In two remarkable papers, [5] and [4], building in particular on work of Zelditch ([33, 34, 35, 36, 37, 38]) and Folland and Stein ([8, 9]), Chang and Rabinowitz have recently made groundbreaking progress in the study of the scaling asymptotics of (1.2), and considerably clarified the analogy with the scaling asymptotics of the equivariant Szegö kernels in the line bundle setting. Their construction has been reviewed and somewhat refined in [23], where the focus was on the near-diagonal case. Here we report on work in progress extending the results of [23] to the equivariant context; in the action-free case, our treatment includes the near-graph setting considered in [4].

Let G be compact connected Lie group G of dimension d_G, with Lie algebra $\mathfrak{g}$, coalgebra $\mathfrak{g}^\vee$, and unitary dual $\widehat{G}$; for every irreducible representation $\nu \in \widehat{G}$ we shall denote by $\Xi_\nu : G \to \mathbb{C}$ its character and by $\dim(\nu)$ the dimension of its total space.

Let be given an isometric action $\mu : G \times M \to M$ on (M, κ). Then there is an extension $\widetilde{\mu} : G \times \widetilde{M} \to \widetilde{M}$ where G acts on $(\widetilde{M}, J)$ as a group of biholomorphisms. This extended holomorphic action preserves ρ and Ω, and is intertwined by the imaginary time exponential with the cotangent action on $T^\vee M \cong TM$. Hence $\widetilde{\mu}$ is Hamiltonian for Ω, with a moment map $\Phi : \widetilde{M} \to \mathfrak{g}^\vee$ intertwined with the standard cotangent moment map.

For every $\tau \in (0, \tau_{\max})$, $\widetilde{\mu}$ restricts to an action $\mu^\tau : G \times X^\tau \to X^\tau$ by CR contact automorphisms commuting with the flow of $\upsilon^\tau_{\sqrt{\rho}}$. This determines unitary representations of G on $H(X^\tau)$ and preserving each eigenspace V_j; by the Theorem of Peter and Weyl, there are unitary equivariant decompositions

$$H(X^\tau) = \bigoplus_{\nu \in \widehat{G}} H(X^\tau)_\nu, \qquad V_j = \bigoplus_{\nu \in \widehat{G}} V_{j,\nu}, \qquad H(X^\tau)_\nu = \bigoplus_{j=1}^{+\infty} V_{j,\nu} \tag{1.3}$$

where $H(X^\tau)_\nu$ is the ν-th equivariant component of $H(X^\tau)$, and $V_{j,\nu} = V_j \cap H(X^\tau)_\nu$.

If $\Pi^\tau_{j,\nu} : L^2(X^\tau) \to V_{j,\nu}$ is the orthogonal projection, the ν-th equivariant analogue of (1.2) is the smoothing operator kernel

$$\Pi^\tau_{\chi,\nu,\lambda} := \sum_j \widehat{\chi}(\lambda - \lambda_j)\, \Pi^\tau_{j,\nu} \in \mathscr{C}^\infty(X^\tau \times X^\tau), \tag{1.4}$$

which probes into the asymptotic distribution of the eigenvalues of $\mathfrak{D}^\tau_{\sqrt{\rho}}$ restricted to $H(X^\tau)_\nu$ and of the asymptotic concentration behaviour of the corresponding equivariant eigenfunctions (belonging to $V_{j,\nu}$).

A first part of the results we are reporting on concerns the scaling asymptotics of (1.4).

A parallel issue, specific to the Grauert tube setting, involves the complexified eigenfunctions of the non-negative Laplacian Δ of (M, κ). Let $0 = \mu_1^2 < \mu_2^2 < \cdots \uparrow +\infty$, where $\mu_j \geq 0$, be the distinct eigenvalues of Δ, and let $W_j \subseteq L^2(M)$ be the (finite-dimensional) eigenspace of μ_j^2. The induced unitary representation of G on $L^2(M)$ (defined in terms of the invariant Riemannian density on M) determines unitary equivariant decompositions

$$L^2(M) = \bigoplus_{\nu \in \widehat{G}} L^2(M)_\nu, \qquad W_j = \bigoplus_{\nu \in \widehat{G}} W_{j,\nu}, \qquad L^2(M)_\nu = \bigoplus_{j=1}^{+\infty} W_{j,\nu}, \tag{1.5}$$

where $W_{j,\nu} = W_j \cap L^2(M)_\nu$. For every j, let us choose an orthonormal basis $(\varphi_{j,\nu,k})_k$ of $W_{j,\nu}$. By a fundamental observation of Boutet de Monvel [1], there exists $\tau_0 \in (0, \tau_{\max}]$ such that any eigenfunction φ of Δ admits a holomorphic extension $\widetilde{\varphi}$ to $\widetilde{M}^{\tau_0}$; furthermore, by suitably rescaling the complexified eigenfunctions $(\widetilde{\varphi}_{j,\nu,k})_k$ one obtains, by restriction to X^τ, a Riesz basis of $H(X^\tau)$ (for discussions and alternative arguments, see [12, 14, 15, 29, 30, 34, 38]). The asymptotic study of the complexified eigenfunctions, also initiated by Zelditch (see, e.g., [33, 34, 36]), in the present equivariant setting is pivoted on the study of the following analogue of (1.4):

$$P^\tau_{\chi,\nu,\lambda}(x, y) := \sum_j \widehat{\chi}(\lambda - \mu_j)\, e^{-2\tau\mu_j} \sum_k \widetilde{\varphi}_{j,\nu,k}(x)\, \overline{\widetilde{\varphi}_{j,\nu,k}(y)}. \tag{1.6}$$

To motivate the exponential 'tempering factor' $e^{-2\tau\mu_j}$ in (1.6), let us recall that the relation between an eigenfunction φ of Δ for an eigenvalue μ^2 and its complexification $\tilde{\varphi}$ (restricted to X^τ) is mediated by the so-called Poisson-wave operator $P^\tau : \mathscr{C}^\infty(M) \to$

$\mathscr{O}(X^\tau)$ (the latter being the ring of CR-holomorphic functions on X^τ), according to the relation

$$P^\tau(\varphi) = e^{-\tau\mu}\,\widetilde{\varphi}. \tag{1.7}$$

The composition $U_{\mathbb{C}}(2\,\imath\,\tau) := P^\tau \circ P^{\tau\,*}$ is a self-adjoint Fourier integral operator with complex phase on X^τ, having the same complex canonical relation as Π^τ. In the action free case (that is, with no ν involved), one can regard (1.6) as a smoothing of the distributional kernel of $U_{\mathbb{C}}(2\,\imath\,\tau)$, pertaining to a spectral band drifting to infinity as $\lambda \to +\infty$; in the general equivariant case, one needs only compose with the projection with the ν-th isotypical component. We refer to the derivation in (2.6)-(2.10) below for more details. Furthermore, with notation as in (1.7), by [34, Corollary 3] for suitable constants $C_j > 0$ and any eigenvalue $\mu \gg 0$ one has estimates

$$C_1\,\mu^{-\frac{d-1}{2}}\,e^{\tau\mu} \le \sup\left\{\left|\widetilde{\varphi}(x)\right| : x \in X^\tau\right\} \le C_2\,\mu^{\frac{d+1}{4}}\,e^{\tau\mu};$$

thus, $e^{-2\tau\mu_j}$ is exactly the factor needed in (1.6) in order to temper the exponential growth of the complexified eigenfunctions.

In this announcement, we will outline some results concerning the local scaling asymptotics of (1.4) and (1.6) as $\lambda \to +\infty$, and hint at some applications; more complete statements and detailed proofs will appear in [10]. All the results discussed in this work rest on the following general hypothesis.

Assumption 1.1. We shall assume that the following holds:

(1) $\Phi^{-1}(0) \cap (\widetilde{M} \setminus M) \neq \emptyset$;
(2) G acts locally freely on the latter set, i.e., $0 \in \mathfrak{g}^\vee$ is a regular value of $\Phi|_{\widetilde{M}\setminus M}$;
(3) $\mathrm{supp}(\chi) \subseteq (t_0 - \epsilon, t_0 + \epsilon)$ for some $t_0 \in \mathbb{R}$ and $\epsilon > 0$, with ϵ suitably small.

Let us set

$$Z := \Phi^{-1}(0), \qquad Z^\tau := Z \cap X^\tau.$$

Thus $Z^\tau \neq \emptyset$ and G acts locally freely on Z^τ. This implies $d > d_G$.

Furthermore, since μ^τ commutes with the 'geodesic flow' Γ^τ, i.e.,

$$\mu^\tau_g \circ \Gamma^\tau_t = \Gamma^\tau_t \circ \mu^\tau_g \quad \forall\, g \in G,\; t \in \mathbb{R},$$

there is an induced $G \times \mathbb{R}$-action on X^τ. If $x \in X^\tau$, let us pose

$$x^{G\times\chi} := \left\{\mu^\tau_g \circ \Gamma^\tau_t(x) : g \in G,\; t \in \mathrm{supp}(\chi)\right\},$$

and

$$\mathfrak{X}^\tau_\chi := \left\{(x_1, x_2) \in Z^\tau \times Z^\tau : x_1 \in x_2^{G\times\chi}\right\}.$$

Then (1.4) and (1.6) are negligible outside a shrinking neighbourhood of $\mathfrak{X}^\tau_\chi$ in $X^\tau \times X^\tau$.

Theorem 1.1. *For any $C, \epsilon' > 0$, we have*

$$\Pi^\tau_{\chi,\nu,\lambda}(x, y) = O\left(\lambda^{-\infty}\right) \quad \textit{and} \quad P^\tau_{\chi,\nu,\lambda}(x, y) = O\left(\lambda^{-\infty}\right)$$

uniformly for

$$\max\left\{\mathrm{dist}_{X^\tau}\left(x, y^{G\times\chi}\right), \mathrm{dist}_{X^\tau}\left(x, Z^\tau\right)\right\} \ge C\,\lambda^{\epsilon'-1/2}.$$

One is thus led to consider scaling asymptotics at points $(x_1, x_2) \in \mathfrak{X}^\tau_\chi$; these scaling asymptotics are conveniently expressed in special systems of local coordinates on X^τ, called *normal Heisenberg local coordinates* in [23] (in the following, we shall use the acronym NHLCs); the latter are slight specializations of the Heisenberg local coordinates introduced in this setting in [5, 4], borrowing from constructions of Folland and Stein in [8, 9] (we refer to [23, §3] for precise definitions and a detailed discussion). In such a system of coordinates, ψ^τ (the phase of the Fourier integral operator describing Π^τ) has a canonical form, which allows for explicit computations.

Given a system of NHLCs centered at $x \in X^\tau$, we shall denote a point with coordinates $(\theta, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^{2d-2}$ in additive notation $x + (\theta, \mathbf{v})$. Displacement in θ is tangent to first order to the orbits of the geodesic flow, and in $\mathbf{v}$ to the CR distribution in X^τ. When $\theta = 0$, we shall simply write $x + \mathbf{v}$. Slightly more precisely, let us consider the direct sum decomposition of vector bundles

$$TX^\tau = \mathcal{T}^\tau \oplus \mathcal{H}^\tau,$$

where $\mathcal{T}^\tau = \mathrm{span}(v^\tau_{\sqrt{\rho}})$ is the 'vertical tangent bundle' and $\mathcal{H}^\tau = \ker(\alpha^\tau)$ is the 'horizontal tangent bundle' (this heuristic terminology is inspired by the line bundle setting). Given a system of NHLCs centered at x, we have $\partial/\partial\theta|_x \in \mathcal{T}^\tau_x$, and $\partial/\partial\mathbf{v}|_x \in \mathcal{H}^\tau_x$ for every $\mathbf{v}$.

Regarding the scaling asymptotics of (1.4), let us mention here two notable special cases, and refer to [10] for a more complete discussion. First, let us focus on the action-free case, which was treated by Chang and Rabinowitz in [4]. Our results in this case refine those in [4].

When G is trivial, we shall write x^χ for $x^{G\times\chi}$, and $\Pi^\tau_{\chi,\lambda}(x,y)$ for $\Pi^\tau_{\chi,\nu,\lambda}(x,y)$. If $\chi \in \mathcal{C}^\infty_c\big((t_0-\epsilon, t_0+\epsilon)\big)$ for some arbitrary $t_0 \in \mathbb{R}$ and $\epsilon > 0$ sufficiently small, then for any $x_1 \in x_2^\chi$ there is a unique $t_1 = t_1(x_1,x_2) \in \mathrm{supp}(\chi)$ such that $x_1 = \Gamma^\tau_{t_1}(x_2)$. Given choices of NHLCs at x_1 and x_2, there are built-in unitary isomorphisms

$$\mathcal{H}^\tau_{x_j} \cong \mathbb{C}^{d-1} \cong \mathbb{R}^{2d-2},$$

where the latter identification is the standard one; furthermore, there is a uniquely determined symplectic matrix $M_{t_1} \in \mathrm{Sp}(2d-2)$ such that

$$\Gamma^\tau_{-t_1}\big(x_1 + (\theta,\mathbf{v})\big) = x_2 + \big(\theta + R_3(\theta,\mathbf{v}),\, M_{t_1}\mathbf{v} + \mathbf{R}_2(\theta,\mathbf{v})\big),$$

where R_j (respectively, $\mathbf{R}_j$) denotes a generic real-valued (respectively, vector-valued) smooth function vanishing to j-th order at the origin [10]. The scaling asymptotics of $\Pi^\tau_{\chi,\lambda}$ at (x_1,x_2) involve a complex-valued real quadratic function on $\mathcal{H}^\tau_{x_1} \times \mathcal{H}^\tau_{x_2}$, hence on $\mathbb{R}^{2d-2} \times \mathbb{R}^{2d-2}$, depending on M_{t_1}.

To define the latter, given a general $M \in \mathrm{Sp}(2d-2)$ let us define the complex matrix

$$\widetilde{M} := \mathcal{W}\, M \mathcal{W}^{-1}, \qquad \text{where} \qquad \mathcal{W} := \frac{1}{\sqrt{2}}\begin{pmatrix} I_{d-1} & \imath I_{d-1} \\ I_{d-1} & -\imath I_{d-1}\end{pmatrix}.$$

Then

$$\widetilde{M} = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}. \tag{1.8}$$

for certain complex matrices P and Q, where P satisfies $\|P(\mathbf{v})\| \geq \|\mathbf{v}\|$, $\forall\, \mathbf{v} \in \mathbb{C}^{d-1}$ (see [7, §4.1]).

Definition 1.1. Given $M \in \mathrm{Sp}(2d-2)$, let us define $\Psi_M : \mathbb{R}^{2d-2} \times \mathbb{R}^{2d-2} \to \mathbb{C}$ as follows. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{2d-2}$ correspond to $Z_1, Z_2 \in \mathbb{C}^{d-1}$. Then

$$\Psi_M(\mathbf{v}_1,\mathbf{v}_2) := \frac{1}{2}\Big(Z_1^\dagger\, \overline{Q}\, P^{-1} Z_1 + 2\,\overline{Z}_2^\dagger P^{-1} Z_1 - \overline{Z}_2^\dagger P^{-1} Q\, \overline{Z}_2 - \|Z_1\|^2 - \|Z_2\|^2\Big).$$

We refer to [7, 38, 39] for a thorough discussion of the relation of Ψ_M to the metaplectic representation.

Let be given systems of NHLCs on X^τ centered at x_j, $j = 1,2$, and tangent vectors $(\theta_j, \mathbf{v}_j) \in \mathbb{R} \times \mathbb{R}^{2d-2} \cong T_{x_j} X^\tau$. Let us set

$$x_{j,\lambda} := x_j + \left(\frac{\theta_j}{\sqrt{\lambda}}, \frac{\mathbf{v}_j}{\sqrt{\lambda}} \right) \qquad (\lambda > 0).$$

In the following statements, $\sim$ means 'has the same asymptotics as'.

Theorem 1.2. *Suppose that $x_1 = \Gamma^\tau_{t_1}(x_2)$ with $t_1 \in \mathrm{supp}(\chi)$ and choose $C > 0$ and $\epsilon' \in (0, 1/6)$. Then, uniformly for $(\theta_j, \mathbf{v}_j) \in \mathbb{R} \times \mathbb{R}^{2d-2} \cong T_{x_j} X^\tau$ with $\|(\theta_j, \mathbf{v}_j)\| \le C\lambda^{\epsilon'}$ the following asymptotic expansions holds for $\lambda \to +\infty$:*

$$\begin{aligned} &\Pi^\tau_{\chi,\lambda}(x_{1,\lambda}, x_{2,\lambda}) \\ &\sim \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{\lambda}{2\pi\tau} \right)^{d-1} \cdot \exp\left(\frac{1}{\tau} \left[\imath\sqrt{\lambda}(\theta_1 - \theta_2) + \Psi_{M^{-1}_{t_1}}(\mathbf{v}_1, \mathbf{v}_2) \right] \right) \cdot e^{-\imath\lambda t_1} \\ &\quad \cdot e^{\imath\theta^\tau_{t_1}(x)} \cdot \left[\chi(t_1) + \sum_{k \ge 1} \lambda^{-k/2} F_k(x_1, x_2; \theta_1, \mathbf{v}_1, \theta_2, \mathbf{v}_2) \right], \end{aligned}$$

$$\begin{aligned} &P^\tau_{\chi,\lambda}(x_{1,\lambda}, x_{2,\lambda}) \\ &\sim \frac{1}{\sqrt{2\pi}} \cdot \left(\frac{1}{2} \right)^{d-1} \cdot \left(\frac{\lambda}{\pi\tau} \right)^{\frac{d-1}{2}} \cdot \exp\left(\frac{1}{\tau} \left[\imath\sqrt{\lambda}(\theta_1 - \theta_2) + \Psi_{M^{-1}_{t_1}}(\mathbf{v}_1, \mathbf{v}_2) \right] \right) \\ &\quad \cdot e^{\imath\tilde{\theta}^\tau_{t_1}(x)} \cdot e^{-\imath\lambda t_1} \cdot \left[\chi(t_1) + \sum_{k \ge 1} \lambda^{-k/2} \widetilde{F}_k(x_1, x_2; \theta_1, \mathbf{v}_1, \theta_2, \mathbf{v}_2) \right], \end{aligned}$$

where $e^{\imath\theta^\tau_{t_1}(x)}$, $e^{\imath\tilde{\theta}^\tau_{t_1}(x)}$ are appropriate unitary factor depending only on x_1 and x_2, and $F_k(x_1, x_2; \cdot)$, $\widetilde{F}_k(x_1, x_2; \cdot)$ are polynomials in the rescaled variables of degree $\le 3k$ and parity k. Furthermore, these asymptotic expansions hold uniformly along $\mathfrak{X}^\tau$.

One recovers the near-graph asymptotic expansion in [4] (with an explicit determination of the leading order factor and a bound on the degree of the F_k's) by rescaling according to Heisenberg type, that is, by considering displacements of the type $x_j + (\theta_j/\lambda, \mathbf{v}_j/\sqrt{\lambda})$ with $(\theta_j, \mathbf{v}_j)$ fixed.

Let us now consider the general equivariant case; for the sake of brevity, in this announcement we shall only describe scaling asymptotics along normal directions to Z^τ; more general statements will be given in [10].

Suppose that $\epsilon > 0$ is sufficiently small and $\chi \in \mathscr{C}^\infty_c\big((t_0 - \epsilon, t_0 + \epsilon)\big)$ for some $t_0 \in \mathbb{R}$. Then for any $(x_1, x_2) \in \mathfrak{X}^\tau_\chi$ there is a unique $t_1 = t_1(x_1, x_2) \in \mathrm{supp}(\chi)$ such that $x_1 = \mu^\tau_g \circ \Gamma^\tau_{t_1}(x_2)$ for some $g \in G$. Furthermore, if the stabilizer subgroup of x_1 in G has cardinality $r = r_{x_1}$, then there are exactly r elements $g_1, \ldots, g_r \in G$ such that $x_1 = \mu^\tau_{g_j} \circ \Gamma^\tau_{t_1}(x_2)$.

Fix, as above, systems of NHLCs on X^τ centered at x_j, $j = 1,2$, and tangent vectors $\mathbf{v}^t_j \in T_{x_j} X^\tau$ normal to Z^τ with respect to the Kähler metric; here the superscript t stands for 'transverse', and refers to the notation in [10]. For $\lambda > 0$, let us set

$$x_{j,\lambda} := x_j + \frac{\mathbf{v}^t_j}{\sqrt{\lambda}}.$$

Theorem 1.3. *Under the previous assumptions, as $\lambda \to +\infty$,*

$$\Pi^\tau_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda}) \sim \sum_{\ell=1}^{r} \Pi^\tau_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda})_\ell$$

and

$$P^\tau_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda}) \sim \sum_{\ell=1}^{r} P^\tau_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda})_\ell,$$

where, for each $\ell = 1, \ldots, r$,

$$\begin{aligned}
&\Pi^{\tau}_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda})_{\ell} \\
&\quad \sim \quad e^{-\imath\lambda t_1 - \frac{1}{\tau}\left(\|\mathbf{v}_1^t\|^2 + \|\mathbf{v}_2^t\|^2\right)} \frac{\chi(t_1)}{\sqrt{2\pi}} \left(\frac{\lambda}{2\pi\tau}\right)^{d-1-d_G/2} \dim(\nu) \\
&\quad \cdot \left[\frac{B_{t_1}(x_1, x_2)}{V_{\mathrm{eff}}(x_1)} \cdot \overline{\Xi_\nu(g_\ell)} + \sum_{k\geq 1} \lambda^{-k/2} F_{k,l}\left(x_1, x_2; \mathbf{v}_1^t, \mathbf{v}_2^t\right)\right],
\end{aligned}$$

$$\begin{aligned}
&P^{\tau}_{\chi,\nu,\lambda}(x_{1,\lambda}, x_{2,\lambda})_{\ell} \\
&\sim \quad e^{-\imath\lambda t_1 - \frac{1}{\tau}\left(\|\mathbf{v}_1^t\|^2 + \|\mathbf{v}_2^t\|^2\right)} \frac{\chi(t_1)}{\sqrt{2\pi}} \left(\frac{1}{2}\right)^{d-1-d_G/2} \left(\frac{\lambda}{\pi\tau}\right)^{(d-1-d_G)/2} \\
&\quad \cdot \dim(\nu) \left[\frac{\widetilde{B}_{t_1}(x_1, x_2)}{V_{\mathrm{eff}}(x_1)} \cdot \overline{\Xi_\nu(g_\ell)} + \sum_{k\geq 1} \lambda^{-k/2} \widetilde{F}_{k,l}\left(x_1, x_2; \mathbf{v}_1^t, \mathbf{v}_2^t\right)\right],
\end{aligned}$$

where $B_{t_1}(x_1, x_2)$ and $\widetilde{B}_{t_1}(x_1, x_2)$ depend on Poincaré type data, $V_{\mathrm{eff}}(x_1)$ is the so-called effective potential at x_1 (the volume of the G-orbit through x_1), and $F_{k,l}(x_1, x_2; \cdot, \cdot)$ and $\widetilde{F}_{k,l}(x_1, x_2; \cdot, \cdot)$ are polynomials of degree $\leq 3k$ and parity k.

For an explicit description of $B_{t_1}(x_1, x_2)$ and $\widetilde{B}_{t_1}(x_1, x_2)$ we refer to [10].

Let us mention a few applications of the previous equivariant scaling asymptotics, inspired by the work of Zelditch [34, 38] and Chang and Rabinowitz [4].

An almost immediate consequence of the scaling asymptotics for $P^{\tau}_{\chi,\nu,\lambda}$ is a pointwise bound on the equivariant complexified eigenfunctions: for some constant $C_\nu > 0$,

$$\sup_{x\in X^\tau} \left|\widetilde{\varphi}_{j,\nu,k}(x)\right|^2 \leq C_\nu\, e^{2\tau\mu_j} \left(\frac{\mu_j}{\tau}\right)^{(d-1-d_G)/2},$$

which refines the upper bound in [38, (4)].

Essentially by performing a Gaussian integration on the asymptotics for $P^{\tau}_{\chi,\nu,\lambda}$ one also obtains an estimate on the L^2-norms of the complexified eigenfunctions:

$$\sum_{\lambda\leq\mu_j\leq\lambda+1} \sum_{k_j} \left\|\widetilde{\varphi}_{j,\nu,k_j}\right\|_{L^2(X^\tau)} \leq C^{\tau}_{\nu}\, e^{2\tau\lambda}\, \lambda^{\frac{d-1}{2} - d_G}. \tag{1.9}$$

We shall also briefly report on an equivariant version of the operator norm estimates established by Chang and Rabinowitz in [4]. Namely, we seek an estimate on the norm of $\Pi^{\tau}_{\chi,\nu,\lambda}$, viewed as an operator $L^p(X^\tau) \to L^q(X^\tau)$.

In the action-free case, Chang and Rabinowitz have given the estimate

$$\left\|\Pi^{\tau}_{\chi,\lambda}\right\|_{L^p\to L^q} \leq C^{\tau}_{p}\, \lambda^{(d-1)\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{1.10}$$

for $p, q \geq 1$. This is the analogue of a similar operator norm estimate by Shiffman and Zelditch on the equivariant components of the Szegő kernel in the line bundle setting [26]; the proof follows the same lines and is based on pairing scaling asymptotics with the Shur–Young inequality [28].

In the equivariant case, (1.10) admits the following refinement.

Theorem 1.4. *Under Assumption 1.1, for some $C_{p,\nu} > 0$ we have*

$$\left\|\Pi^{\tau}_{\chi,\nu,\lambda}\right\|_{L^p\to L^q} \leq C_{p,\nu} \left(\frac{\lambda}{\tau}\right)^{\left(d-1-\frac{1}{2}d_G\right)\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Similar estimates hold for $P^{\tau}_{\chi,\nu,\lambda} : L^p(X^\tau) \to L^q(X^\tau)$. The proof is essentially an equivariant adaptation of the one for the action-free case; a preliminary step is to ensure that the scaling asymptotics for $\Pi^{\tau}_{\chi,\nu,\lambda}$ and $P^{\tau}_{\chi,\nu,\lambda}$ hold uniformly along Z^τ.

These operator norm estimates may also be viewed as the counterpart, in the Grauert tube setting, of estimates proved by Sogge in the real domain for the spectral projector of the Laplacian relative to a spectral band drifting to infinity ([27, 28]). While Sogge's estimates are rarely sharp on general manifolds, in the action free case Chang and Rabinowitz have proved in [4] that the corresponding estimates in the complex domain are generally sharp. Actually, sharpness may also be established in the present equivariant setting, by adapting the argument in [4] to the functions $\Phi^{y}_{\chi,\nu,\lambda} : X^\tau \to \mathbb{C}$ given by

$$\Phi^{y}_{\chi,\nu,\lambda}(\cdot) := \frac{\Pi^{\tau}_{\chi,\nu,\lambda}(\cdot, y)}{\|\Pi^{\tau}_{\chi,\nu,\lambda}(\cdot, y)\|_{L^p}},$$

where $y \in Z^\tau$ is fixed, and making use of the equivariant asymptotics in Theorem 1.3.

For a more precise discussion of the above, and for additional applications, such as equivariant Husimi and Weyl type estimates, see [10].

2. Strategy of proof

Let us give a broad illustration of the basic strategy of the proofs in [10] of Theorems 1.1, 1.2, and 1.3, focusing first on the Toeplitz operator $\mathfrak{D}^{\tau}_{\sqrt{\rho}}$.

To begin with, if $P_\nu : L^2(X^\tau) \to L^2(X^\tau)_\nu$ is the orthogonal projector onto the ν-th isotypical component, then

$$\Pi^{\tau}_{\chi,\nu,\lambda} = P_\nu \circ \Pi^{\tau}_{\chi,\lambda}. \tag{2.1}$$

In terms of distributional kernels, this reads

$$\Pi^{\tau}_{\chi,\nu,\lambda}(x_1, x_2) = \dim(\nu) \int_G \overline{\Xi_\nu(g)}\, \Pi^{\tau}_{\chi,\lambda}\left(\mu^{\tau}_{g^{-1}}(x_1), x_2\right) \mathrm{d}V_G(g). \tag{2.2}$$

Next, one considers the 1-parameter group of unitary Toeplitz operators

$$U^{\tau}_{\sqrt{\rho}}(t) := e^{\imath\, t\, \mathfrak{D}^{\tau}_{\sqrt{\rho}}} \qquad (t \in \mathbb{R})$$

generated by $\mathfrak{D}^{\tau}_{\sqrt{\rho}}$. Then (passing to distributional kernels)

$$\Pi^{\tau}_{\chi,\lambda}(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\imath\lambda t}\, \chi(t)\, U^{\tau}_{\sqrt{\rho}}(t; x, y)\, \mathrm{d}t. \tag{2.3}$$

The next step, following an insight of Zelditch which goes back to [32] (see also [34, 38, 39]), is to represent $U^{\tau}_{\sqrt{\rho}}(t)$, up to a smoothing operator, as a 'dynamical Toeplitz operator':

$$U^{\tau}_{\sqrt{\rho}}(t) = \Pi^\tau \circ P^{\tau}_t \circ \Pi^{\tau}_{-t} + S^{\tau}_t, \tag{2.4}$$

where $\Pi^{\tau}_{-t}(x, y) := \Pi^\tau\left(\Gamma^{\tau}_{-t}(x), y\right)$, P^{τ}_t is an appropriate smoothly varying zeroth-order pseudo-differential operator, and S^{τ}_t is a smoothly varying smoothing operator.

In order to make the previous composition asymptotically computable, one needs to insert the microlocal description of Π^τ a Fourier integral operator from [2]:

$$\Pi^\tau(x, y) \sim \int_0^{+\infty} e^{\imath\, v\, \psi^\tau(x,y)}\, s^\tau\left(x, y, v\right) \mathrm{d}v,$$

where the phase ψ^τ is determined by the defining equation of $X^\tau \subseteq \widetilde{M}$, and s^τ is a semiclassical symbol. Up to a negligible contribution, one obtains for $\Pi^\tau_{\chi,\nu,\lambda}(x_1,x_2)$ an expression as an oscillatory integral, with a phase factor of the form $e^{\imath\,\lambda\,\Psi}$, where

$$\Psi(x_1,x_2;g,t,y,u,v) := u\,\psi^\tau\big(\mu^\tau_{g^{-1}}(x_1),y\big) + v\,\psi^\tau\big(\Gamma^\tau_{-t}(y),x_2\,y\big) - t. \tag{2.5}$$

The next step is to use integration by parts to justify the reduction of integration to a shrinking domain, which can be parametrized using rescaled coordinates. Working in NLHC's and then rescaling, one can expand Ψ building on the computations in [23, §3.4]. The outcome is a reformulation of (2.3), up to a negligible contribution, as an oscillatory integral in the parameter $\sqrt{\lambda} \to +\infty$, with an explicit simple real phase. One then proceeds essentially applying the Stationary Phase Lemma.

Let us sketch how the previous approach can be modified to deal with the 'complexified smoothed spectral projector' (1.6) (see the discussions in [34, 38, 5, 4]). The natural starting point is the half-wave operator $U(t) := e^{\imath\,t\sqrt{\Delta}} : L^2(M) \to L^2(M)$; assuming that the orthonormal basis $(\varphi_{j,k})_k$ of V_j is composed of real functions, its distributional kernel is

$$U(t;m,n) := \sum_{j=1}^{+\infty} e^{\imath\,\mu_j\,t} \sum_k \varphi_{j,k}(m)\cdot\varphi_{j,k}(n) \qquad (m,n\in M). \tag{2.6}$$

Its equivariant version is the 1-parameter group of unitary operators $U_\nu(t) := P_\nu \circ e^{\imath\sqrt{\Delta}} : L^2(M)_\nu \to L^2(M)_\nu$, which may be naturally regarded as a 1-parameter family of operators $U_\nu(t) : L^2(M) \to L^2(M)$. Its distributional kernel is

$$U_\nu(t;m,n) := \sum_{j=1}^{+\infty} e^{\imath\,\mu_j\,t} \sum_k \varphi_{j,\nu,k}(m)\cdot\varphi_{j,\nu,k}(n), \tag{2.7}$$

where the inner sum is now over an orthonormal (real) basis $(\varphi_{j,\nu,k})_k$ of $V_{j,\nu}$.

For $\tau > 0$, the kernel $U(t;m,n)$ admits a holomorphic extension in the time variable, denoted $U(t+\imath\,\tau,m,n)$, given by

$$U(t+\imath\,\tau;m,n) := \sum_{j=1}^{+\infty} e^{(-\tau+\imath\,t)\,\mu_j} \sum_k \varphi_{j,k}(m)\cdot\varphi_{j,k}(n). \tag{2.8}$$

When $t=0$, in particular, $U(\imath\,\tau;\cdot,\cdot)$ is the Poisson kernel on $M\times M$. For $\tau>0$, the Poisson kernel is real-analytic on $M\times M$, and for $\tau\in(0,\tau_0)$ it can be extended in the first spacial variable to a kernel

$$P^\tau(x,n) := \sum_{j=1}^{+\infty} e^{-\tau\,\mu_j} \sum_k \widetilde{\varphi}_{j,k}(x)\cdot\varphi_{j,k}(n) \qquad (x\in X^\tau,\ n\in M), \tag{2.9}$$

which defines a Fourier integral operator P^τ with complex phase and of degree $-(d-1)/4$. For every $s\in\mathbb{R}$, (2.9) determines an isomorphism $P^\tau : \mathscr{W}^s(M) \to \mathscr{O}^{s+\frac{d-1}{4}}(X^\tau)$, where $\mathscr{W}^s(M)$ is the s-th Sobolev space on M, and $\mathscr{O}^{s+\frac{d-1}{4}}(X^\tau)$ is the space of CR functions on X^τ that are in $\mathscr{W}^{s+\frac{d-1}{4}}(X^\tau)$. The distributional kernel of $P^\tau_\nu := P^\tau\circ P_\nu$ is then clearly

$$P^\tau_\nu(x,n) := \sum_{j=1}^{+\infty} e^{-\tau\,\mu_j} \sum_k \widetilde{\varphi}_{j,\nu,k}(x)\cdot\varphi_{j,\nu,k}(n). \tag{2.10}$$

Following Zelditch, one considers the composition

$$U_{\mathbb{C}}(t+2\,\imath\,\tau) := P^\tau\circ U(t)\circ P^{\tau\,*}, \tag{2.11}$$

which is a Fourier integral operator with complex phase on X^τ of degree $-(d-1)/2$, and its equivariant version

$$U_{\mathbb{C}}(t+2\,\imath\,\tau)_\nu := P^\tau_\nu\circ U(t)\circ P^{\tau\,*} = P^\tau\circ U(t)\circ P^{\tau\,*}_\nu. \tag{2.12}$$

The link between (1.6) and (2.12) is encapsulated in the relation

$$\begin{aligned}(2.13)\qquad P^{\tau}_{\chi,\nu,\lambda}(x_1,x_2) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\chi(t)\,e^{-\imath\lambda t}\,U_{\mathbb{C}}(t+2\,\imath\,\tau;x,y)_\nu\,\mathrm{d}t\\ &= \frac{\dim(\nu)}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{d}t\int_G \mathrm{d}V_G(g)\\ &\quad \left[\chi(t)\,e^{-\imath\lambda t}\,\Xi_\nu\left(g^{-1}\right)U_{\mathbb{C}}\left(t+2\,\imath\,\tau;\mu^{\tau}_{g^{-1}}(x_1),x_2\right)\right].\end{aligned}$$

Furthermore, $U_{\mathbb{C}}\,(t+2\,\imath\,\tau)$ admits a description via dynamical Toeplitz operators analogous to (2.4): for an appropriate smoothly varying pseudodifferential operator Q^{τ}_t of degree $-(d-1)/2$ on X^{τ}, one has

$$(2.14)\qquad U_{\mathbb{C}}\,(t+2\,\imath\,\tau) = \Pi^{\tau}\circ Q^{\tau}_t\circ\Pi^{\tau}_{-t}+R^{\tau}_t,$$

where R^{τ}_t is a smoothly varying smoothing operator.

The rest of the argument is formally similar to the one for $\Pi^{\tau}_{\chi,\nu,\lambda}$.

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