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Posets and fractional Calabi–Yau categories

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(Recommended by Boris Hasselblatt)

Abstract. This article deals with a relationship between derived categories of modules over some partially ordered sets and triangulated categories arising from quasi-homogeneous isolated singularities. It produces heuristics for the existence of derived equivalences between posets, using the geometric category as an auxiliary intermediate. The notion of Weight plays a central role as a simple footprint of the derived categories under consideration.

To see a World in a Grain of Sand
And a Heaven in a Wild Flower
Hold Infinity in the palm of your hand
And Eternity in an hour
William Blake

Introduction

The aim of this article is to explain a simple idea that starts from the combinatorics of partially ordered sets (posets) and leads to conjectures about fractional Calabi–Yau categories and their triangle-equivalences.

Let us start with an infinite family of combinatorial objects, given as the disjoint union of finite sets P_n indexed, for example, by positive integers. Suppose that for every n, the cardinality of P_n can be written under the specific shape

$$|P_n| = \frac{\prod_{i=1}^{m}(D-d_i)}{\prod_{i=1}^{m} d_i},$$

where m is a positive integer, $d_1,\dots,d_m$ is a multi-set of positive integers, and D is a positive integer, all depending on n in a regular way.

Then, one can hope for the following statement (♣):

There exists a family of partial orders $(P_n,\le)$ such that, for all n, the derived category of $(P_n,\le)$ is triangle-equivalent to the fractional Calabi–Yau category associated with a generic isolated quasi-homogeneous singularity with variable weights $(d_1,\dots,d_m)$ and total weight D.

In this statement, the derived category of a poset $(P,\le)$ means the bounded derived category of finite-dimensional modules over its incidence algebra over a field. The meaning of the category attached to the singularity is some kind of derived Fukaya category, providing a categorified version of the classical Milnor theory of isolated hypersurface

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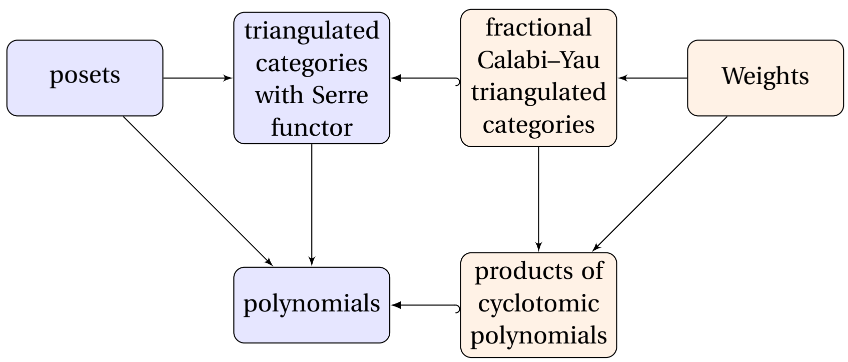
singularities. Milnor theory is briefly recalled in Section 1 and the related categorical framework is considered later in Section 2.

If the property (♣) holds, it has a much more concrete consequence, namely the Coxeter polynomial of the poset $(P_n, \leq)$ gets identified with the characteristic polynomial of the monodromy for the isolated singularity and can therefore be computed by just knowing the integers $d_1, \dots, d_m$ and D, by results of Milnor and Orlik explained in Section 1. The data of $d_1, \dots, d_m$ and D, satisfying the appropriate conditions, will be called a *Weight*.[1]

As the Coxeter polynomial is also easy to compute directly for any given poset (using the Coxeter matrix, see Appendix A), this gives a criterion to check if a family of partial orders on P_n could have the expected property. When the first Coxeter polynomials for small n are as expected, this gives a strong evidence for the statement (♣) above. In this case, let us say that the *Coxeter criterion* holds for this family of posets. This may seem weak, as it is known that the Coxeter polynomial is not a complete invariant of triangle-equivalence. The fact that this holds in family improves the solidity of the Coxeter criterion, as the probability of a random coincidence decrease.

Here is a schematic description of the situation:



The top-left horizontal arrow sends a poset to the derived category of modules over its incidence algebra over a field. This is a many-to-one application, as one can easily find examples of derived-equivalent but not isomorphic posets. The top-right horizontal arrow is the Fukaya–Seidel categorification of the Milnor fiber construction. The vertical arrows from triangulated categories to polynomials are given by the characteristic polynomial of the Auslander–Reiten functor (which is a shifted version of the Serre functor). As said above, there are direct constructions of these polynomials from posets and from Weights, which give the diagonal arrows.

This diagram is expected to be fully compatible with the natural monoidal structures, namely cartesian product of posets, tensor product of triangulated categories, and the monoid structure on Weights introduced in Section 3. For the bottom line, one can use a tensor product of polynomials, but we will not need that. This compatibility is known for the left half of the diagram.

So the Coxeter criterion means that we have a sequence of polynomials coming from the top-left that can be identified with a sequence of polynomials coming from the top-right. The main idea is to take this equality as a rather strong hint that the triangulated categories should be themselves triangle-equivalent.

[1] We write Weight with a capital letter to make it more clear that it stands for a precise technical definition, see Definition 1.

As we will see in the examples below, partial orders on a given combinatorial family that satisfy the Coxeter criterion are not necessarily unique. There can very well be several distinct families of posets with the same cardinalities, all having their derived categories triangle-equivalent to the same singularity category. In this case, the implied derived equivalences between the different posets of the same cardinality can sometimes be proved by other means.

Given a family of combinatorial objects, finding some correct partial orders is not easy in general. Sometimes the most natural partial orders all fail to satisfy the Coxeter criterion. One can then look for more subtle partial orders on the same combinatorial objects, or maybe on other combinatorial objects counted by the same sequence of numbers.

Another implication is that one can use factorizations in the monoid of Weights to propose conjectural derived equivalences. If a Weight is associated with a poset P, and factors into simpler Weights associated with smaller posets $Q_1,\dots,Q_k$, then one should expect a derived equivalence between P and the cartesian product of posets $Q_1\times\cdots\times Q_k$. A very simple case is given by the Dynkin quivers $\mathbb{D}_4$ and $\mathbb{A}_2\times\mathbb{A}_2$.

1. Quasi-homogeneous isolated singularities

The theory of singularities of algebraic functions from $\mathbb{C}^m$ to $\mathbb{C}$ is a very classical topic, with a vast literature. One could cite for instance the famous classification by Arnold of rigid isolated singularities by the Dynkin diagrams of type $\mathbb{ADE}$. Even more well-studied is the case of singularities of quasi-homogeneous algebraic functions, where each coordinate on $\mathbb{C}^m$ is given a specific weight and the function is assumed to be homogeneous for the total degree with respect to these weights.

Let us sketch briefly the celebrated construction of Milnor for isolated singularities. For more details, the reader may consult [43, 17]. Let f be a quasi-homogeneous polynomial function from $\mathbb{C}^m$ to $\mathbb{C}$. Assume that f has an isolated critical point at $0\in\mathbb{C}^m$ above $0\in\mathbb{C}$. Milnor has shown that over a sufficiently small circle S_ε around $0\in\mathbb{C}$, all the fibers of f (intersected with a small ball around $0\in\mathbb{C}^m$) are smooth and diffeomorphic, with the homotopy type of a bouquet of μ_f spheres of dimension $m-1$. The fibers have therefore only one interesting homology group H_{m-1}, of dimension μ_f. This locally-trivial fibration over the circle S_ε is called the *Milnor fibration* of f and μ_f is called the *Milnor number* of the singularity.

By turning once over the circle S_ε and following the cycles using local triviality of the Milnor fibration, one gets a linear endomorphism of the homology group $H_{m-1}(f^{-1}(\varepsilon))$. This is called the *monodromy* of the singularity.

In the case of a quasi-homogeneous polynomial f with an isolated singularity, Milnor and Orlik [44] have given an explicit formula for the characteristic polynomial of the monodromy (or rather for its roots) depending only on the degrees $d_1,\dots,d_m$ of the variables and the total degree D of f. As a special case of this formula, the Milnor number is given by

$$\mu_f=\frac{\prod_{i=1}^{m}(D-d_i)}{\prod_{i=1}^{m}d_i}. \tag{1.1}$$

Let us now present their formula briefly. The following description is a streamlined presentation, with slightly modified notations, of Milnor and Orlik result in [44, §3], see also [43, §9]. Let u_i be the numerator of D/d_i as a reduced fraction, for $i=1,\dots,m$. For each divisor j of D, define

$$\chi_j = \prod_{\substack{i=1 \\ u_i | j}}^{m} \frac{d_i - D}{d_i}. \tag{1.2}$$

Then for each divisor j of D, there is a relative integer s_j such that

$$j s_j = \sum_{d|j} \mu_{j/d} \chi_d, \tag{1.3}$$

where μ is (just here) the standard number-theoretic Möbius function. The characteristic polynomial of the monodromy is then

$$\left(\prod_{j|D} (t^j - 1)^{s_j} \right)^{(-1)^m}. \tag{1.4}$$

Consider, for example, the degrees $(d_1, d_2, d_3) = (2,3,4)$ and $D = 10$. Then $(u_1, u_2, u_3) = (5,10,5)$. One computes that $(\chi_1, \chi_2, \chi_5, \chi_{10}) = (1,1,6,-14)$, and it follows that $(s_1, s_2, s_5, s_{10}) = (1,0,1,-2)$. As a product of cyclotomic polynomials, the characteristic polynomial is therefore $\Phi_2^2 \Phi_5 \Phi_{10}^2$.

1.1. Hodge structure and *q*-Milnor number. This information about the characteristic polynomial can be refined as follows.

Let us consider the formula

$$\frac{\prod_{i=1}^{m} [D - d_i]_q}{\prod_{i=1}^{m} [d_i]_q}, \tag{1.5}$$

where $[d]_q = (q^d - 1)/(q - 1)$ is the q-analogue of an integer d.

For the degrees $(d_1, \ldots, d_m)$ and D of an isolated quasi-homogeneous singularity, it is known that (1.5) is a polynomial in $\mathbb{N}[q]$, whose coefficients are the dimensions of the homogeneous components of the Jacobian algebra of the singularity. This is a classical statement, see [2], [29, Theorem 6.4] or [52, 5.11].

In this situation, the polynomial (1.5) is also closely related to the eigenvalues of the monodromy. As shown by J. Steenbrink [52, 51], the homology group H_{m-1} carries a mixed Hodge structure, compatible with the monodromy. In the case of quasi-homogeneous isolated singularities, the dimensions of the successive quotients of the Hodge filtration can be encoded by the coefficients of a polynomial in q, which according to [52, 5.11] turns out to be (1.5).

The polynomial (1.5) in fact also contains complete information on the multiplicities of the eigenvalues of the monodromy, hence gives an alternative way to access them, different from the Milnor–Orlik method recalled above. It suffices to look at the coefficients of the unique polynomial representative of (1.5) modulo $q^D - 1$ with degree at most $D - 1$. The coefficients, considered as a cyclic sequence, are then the multiplicities of the D-th roots of unity as eigenvalues of the monodromy.

For a more precise account of these aspects, see [29, §5, §6]. We give an example below.

One can note that setting $q = 1$ in (1.5) recovers the expression (1.1) for the Milnor number μ_f. The expression (1.5) will be called the *q-Milnor number* of the singularity.

In the example of degrees $(2,3,4)$ and total degree 10, one finds

$$q^{12} + q^{10} + q^9 + 2q^8 + q^7 + 2q^6 + q^5 + 2q^4 + q^3 + q^2 + 1,$$

which reduces modulo $q^{10}-1$ to

$$q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+2q^2+2,$$

giving the sequence of coefficients $2,0,2,1,2,1,2,1,2,1$ for powers of q in increasing order.

On the other hand, the characteristic polynomial is

$$(t^5+1)^2(t^4+t^3+t^2+t+1),$$

as seen at the end of the previous section. The multiplicities of the ten roots of unity $(e^{2ik\pi/10})_{k=0}^{9}$ are therefore $0,2,1,2,1,2,1,2,1,2$.

Comparing both sequences, one can see that they are indeed the same up to an appropriate cyclic shift.

2. Fractional Calabi–Yau categories

Fractional Calabi–Yau categories were introduced by M. Kontsevich around 1998 [33] as a natural generalization of Calabi–Yau categories, themselves motivated by the properties of coherent sheaves on Calabi–Yau manifolds. Fractional Calabi–Yau categories sometimes appear in the semi-orthogonal decompositions of bounded derived categories of coherent sheaves on algebraic varieties, and in particular Fano varieties, see, for example, [36].

Recall that a *Serre functor* in a triangulated category $\mathscr{T}$ is an auto-equivalence S of $\mathscr{T}$ such that there is a bi-natural isomorphism

$$Hom(X,Y)^* \simeq Hom(Y,SX),$$

where $*$ is the linear dual over the ground field. For more on this notion, we refer to [6, 31]. The existence of a Serre functor S on a triangulated category is equivalent to the existence of an Auslander–Reiten translation functor τ. These two functors are unique up to isomorphism and related by $S=\tau[1]$, where $[1]$ is the shift functor. They both exist, for example, for the bounded derived categories of modules over a finite dimensional algebra of finite global dimension over a field, see [31, §3.1]. This includes incidence algebras of finite posets over a field.

A triangulated category $\mathscr{T}$ is a *fractional Calabi–Yau category* if it has a Serre functor S and there exist integers p and q such that $S^q \simeq [p]$ as functors. Here $[p]$ is the p-th power of the shift functor. In this case, the Calabi–Yau dimension is the pair (p,q), often denoted p/q by a common abuse of notation.

2.1. Fukaya–Seidel categories for singularities. Let us keep the same notations as in Section 1.

Attached to each quasi-homogeneous isolated singularity f, there is a triangulated category $\mathscr{D}_f$ which is a categorification of the Milnor geometric theory described in Section 1. This category has been defined by Seidel [50, 49] in the context of symplectic geometry. It is obtained as the derived category (or homology category) of a A_∞-category of Fukaya type, in a directed version, starting from the Milnor fibration and using a morsification.

These categories $\mathscr{D}_f$ are expected to have the following three properties:

(S0) The Grothendieck group $K_0(\mathscr{D}_f)$ is identified with the homology group H_{m-1}, in such a way that the Auslander–Reiten functor on $\mathscr{D}_f$ induces a linear endomorphism on $K_0(\mathscr{D}_f)$ which is identified with the monodromy (up to an appropriate shift).

(S1) This category $\mathscr{D}_f$ is fractional Calabi–Yau, with Calabi–Yau dimension (C,D), where

$$C = \sum_{i=1}^{m} (D - 2d_i). \tag{2.1}$$

(S2) The construction $f \mapsto \mathscr{D}_f$ is multiplicative, sending the Thom–Sebastiani sum of singularities to the tensor product of triangulated categories. This was explicitly stated as [3, Conjecture 1.3] and is apparently still open.

The first statement seems to be folklore in the domain. To make the appropriate shift precise would require a close examination of the definition of the category. As far as the author can tell, the other properties are known in some cases, but still conjectural in general. Most of the articles on closely related topics refer directly to the original book by Seidel [49] for the definition of the Fukaya–Seidel categories. Some other relevant articles are [24, 25, 27, 26, 22, 3].

Note: One could wonder whether the category $\mathscr{D}_f$ for a generic quasi-homogeneous polynomial determines the Weight. The category certainly determines the eigenvalues of the monodromy operator, and this puts strong constraints on the possible q-Milnor numbers, as explained in Section 1.1.

2.2. About mirror symmetry and *B*-model. Mirror symmetry suggests the existence of mirror singularities for isolated quasi-homogeneous singularities in general. This is known only in some specific cases and has been in particular much studied in the case of invertible polynomials, as considered by Berglund and Hübsch in [5] and by Kreuzer and Skarke in [34]. Then more algebraic methods are available to define and study the same categories, for instance homological matrix factorizations. For a more precise view on this, the reader may see [30], [20, §4], and [21].

In the language of theoretical physics (and with all the necessary caution), the category $\mathscr{D}_f$ has something to do with the A-model for the Landau–Ginzburg potential f, and it should be related to A-branes of this model. The mirror symmetry is supposed to identify these A-branes with B-branes on the mirror manifold, which seem to be better understood in mathematical terms.

3. The monoid of Weights

Let us define in this section a monoid $\mathscr{W}$ whose elements will be called *Weights*. Our notion of Weight is closely related to what is called a *weight system* in singularity theory, see, for example, [29].

Definition 1. *A Weight is a pair $((d_1,d_2,\dots,d_m),D)$ where $d_1,d_2,\dots,d_m$ and D are positive integers such that the formula*

$$\frac{\prod_{i=1}^{m} [D-d_i]_q}{\prod_{i=1}^{m} [d_i]_q} \tag{3.1}$$

defines a polynomial in $\mathbb{N}[q]$. This Weight will be denoted $(d_1,\dots,d_m;D)$.

The order of the d_i is irrelevant. Weights that only differ by multiplying all d_i and D by a common positive integer N are considered to be the same.

Note that $m = 0$ is allowed, as the formula is then the empty product.

One will always assume that $d_i < D - d_i$ for all i, as factors where $2d_i = D$ do not contribute to the product (3.1).

Dividing every d_i and D by their greatest common divisor and then sorting the d_i in increasing order gives a unique canonical representative.

For example, in the case $(2,3;8)$, one finds the fraction

$$\frac{[6]_q[5]_q}{[2]_q[3]_q} = q^6 + q^4 + q^3 + q^2 + 1,$$

so that this is indeed a Weight.

The value at $q = 1$ of the formula (3.1) for a Weight α will be called the *Milnor number* μ_α of the Weight. For example, the Milnor number of $(2,3;8)$ is 5.

The expression (3.1) will be called the *q-Milnor number* of the Weight. Note that it depends on the choice of a representative, but only up to substitution of q by some power of q.

3.1. Variations. One can make several variants of the definition above, some weaker and some stronger.

When the condition on the pair $((d_1, d_2, \ldots, d_m), D)$ in this definition is weakened to require only that the value of (3.1) at $q = 1$ is an integer, this will be called a *weak Weight*.

One could similarly require that the quotient should be a polynomial in $\mathbb{Z}[q]$ with value at $q = 1$ in $\mathbb{N}$. We will not use this intermediate notion. It is not clear if one can find something like this which is not a Weight.

For a given Weight, one can consider a generic polynomial of degree D in variables $x_1, \ldots, x_m$ of degrees $d_1, \ldots, d_m$. In order for such a polynomial to define an isolated hypersurface singularity, a stronger condition must be imposed on the Weight, which can be found, for example, in [28, §2]. Examples of Weights not satisfying this stronger condition, such as $(16,18,21,55;165)$, are displayed in [29, Table 1].

As suggested by a referee, proving this stronger but elementary condition may sometimes be easier than directly showing that the q-Milnor number has coefficients in $\mathbb{N}$ by finding a combinatorial interpretation for these coefficients.

3.2. Product. Let $\mathscr{W}$ be the set of all Weights. The set $\mathscr{W}$ can be endowed with the following binary operation. In terms of singularity theory, this corresponds to the Thom–Sebastiani direct sum of hypersurface singularities.

Let $\alpha = (a_1, \ldots, a_m; A)$ and $\beta = (b_1, \ldots, b_n; B)$ be two Weights. Then one defines

$$\alpha \times \beta = (Ba_1, Ba_2, \ldots, Ba_m, Ab_1, \ldots, Ab_n; AB), \tag{3.2}$$

One can check that this is indeed a Weight. This could also be defined as

$$\alpha \times \beta = (B'a_1, B'a_2, \ldots, B'a_m, A'b_1, \ldots, A'b_n; \operatorname{lcm}(A,B)),$$

where $A' = A/\gcd(A,B)$ and $B' = B/\gcd(A,B)$, which is a simpler representative of the same Weight. One could also define the same operation as disjoint union of the a_i and b_i by assuming without loss of generality that $A = B$.

This defines a commutative and associative product $\times$ on the set $\mathscr{W}$, with unit the empty Weight $(\emptyset;1)$.

For example, $(3,5;20) \times (1;5) = (3,4,5;20)$.

There are a few interesting morphisms from $\mathscr{W}$ to other monoids.

The formula (3.1) evaluated at $q = t^{1/D}$ defines a morphism to the multiplicative monoid of Puiseux polynomials in t.

Similarly, the evaluation of (3.1) at $q = 1$ defines a morphism to the multiplicative monoid $\mathbb{N}$. This is just the Milnor number.

From (2.1), one obtains the formula

$$\frac{\sum_{i=1}^{m}(D-2d_i)}{D} \tag{3.3}$$

for the Calabi–Yau dimension seen as a positive rational number. This defines a morphism to the additive monoid $\mathbb{Q}_{>0}$. This is clear when seeing $\times$ as concatenation of Weights sharing the same D. This quantity could be called the *central charge* of the Weight.[2]

For example, the central charge of $(2,3,5,5;15)$ is 2.

3.3. Factorization and prime Weights. Let us say that a Weight is *prime* if it is not the product of several strictly smaller Weights, namely with smaller number of degrees.

In order to check that a Weight $(d_1,\dots,d_m;D)$ is prime or not, one needs to look for non-empty subsets of the d_i that define a Weight and whose non-empty complement also defines a Weight (keeping the same D). This can be done using the expression of q-integers as products of cyclotomic polynomials Φ_d. In small cases, one can easily check in this way that some Weight is prime.

As a simple example, let us prove that the Weight $(3,4,5,6;15)$ is prime. The factors in the q-Milnor number are, after simplification,

$$\frac{\Phi_2\Phi_4\Phi_6\Phi_{12}}{1} \times \frac{\Phi_{11}}{\Phi_2\Phi_4} \times \frac{\Phi_2\Phi_{10}}{1} \times \frac{\Phi_9}{\Phi_2\Phi_6}. \tag{3.4}$$

Because of Φ_6 in its denominator, the fourth term must be grouped with the first one. Then because of Φ_4 in its denominator, the second term must also be grouped with the first one. Then there remains a Φ_2 in the denominator of the result which is forced to also be grouped with the third term.

Consider now the Weight $(2,4,6,7;18)$ with Milnor number 88. One can check that it can be written both as

$$(1;9)\times(4,6,7;18) \quad \text{and as} \quad (1;3)\times(2,4,7;18), \tag{3.5}$$

where in both factorizations all factors are prime. Something similar happens for the Weight $(3,4,7,10;24)$. It follows that in the monoid $\mathscr{W}$ there is no unique factorization in prime elements.

4. The Catalan family

In this section and the following ones, we consider several examples of families of Weights, starting from a simple case.

The Catalan numbers are defined by the formula

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \tag{4.1}$$

which can be written as

$$c_n = \frac{2n\dots(n+2)}{2\dots n}$$

for $n \geq 1$. This comes from the Weight $(2,3,\dots,n;2n+2)$. In this case, it is known that the q-Milnor number is a polynomial in q with positive coefficients. Indeed, this polynomial is enumerating Dyck paths according to the major index, see [48, St000027] and [40].

For small n, the Weights in this family are $\mathbb{A}_1, \mathbb{A}_2, \mathbb{D}_5, S_{1,0}, \mathbb{D}_7 \times \mathbb{E}_6$, using factorization in $\mathscr{W}$ and the notations in the tables of Section B.

[2] It is one third of the central charge appearing in the related $N = (2,2)$ super-conformal field theory.

The next Weight in this family can still be described using the notion of invertible polynomials, as considered in [5] and classified in [34]: it is the Weight corresponding to the chain type of parameter $(7,2,2,2,3)$.

In general, the Weights in the Catalan family cannot be realized by invertible polynomials in the sense of Berglund–Hübsch. The first impossible case happens for $n = 10$, with Milnor number 16796.

It turns out that there are at least two different families of posets that seem to satisfy the Coxeter criterion.

The first family is made of the Tamari lattices T_n, introduced by Tamari in [54]. They have been studied a lot since then, in particular as a special case of the Cambrian lattices in the theory of cluster algebras, see for instance [45]. The underlying set is the set of planar rooted binary trees with n inner vertices and $n+1$ leaves, endowed with the partial order whose covering relations are rotations. The cardinality of T_n is known to be the Catalan number c_n.

In the case of the Tamari lattices, the Coxeter polynomial has been computed by operadic methods in [11]. It is therefore possible to check the Coxeter criterion for large values of n. In principle, one could hope to prove that this general formula coincides with the formula obtained from the sequence of Weights, although this has never been done to our knowledge.

Moreover, B. Rognerud has proved in [47] that the derived category of T_n is indeed fractional Calabi–Yau, of the expected dimension $(n(n-1), 2n+2)$.

The second family of posets is even simpler. The underlying set is the set D_n of Dyck paths of size n, which are lattice paths of length $2n$ using steps $(+1,+1)$ and $(+1,-1)$, starting from $(0,0)$, ending at $(2n,0)$ and never going strictly below the horizontal axis. The number of such paths is known to be c_n too. The partial order on D_n is defined by one path being always weakly below another path. This defines a distributive lattice. In this case, no general formula is known for the Coxeter polynomials, but one can check by computer that they coincide with those of the Tamari lattices for $n \leq 9$.

All this strongly suggests that the posets T_n and D_n are derived-equivalent, and that both are triangle-equivalent to the same triangulated category of geometric origin associated with an isolated singularity. For this reason, this derived equivalence has been stated as a conjecture in [12].

Recently, some intermediate lattices (named the alt-Tamari lattices) have been introduced in [14], that generalize the previous two families and apparently share the same Coxeter polynomials. Together with S. Ladkani [13], we plan to establish the derived equivalences among these intermediate lattices. This would in particular solve the above conjecture on the derived equivalence of the lattices T_n and D_n.

Let us now briefly talk about closely related posets where the Coxeter criterion seems to hold, for other sequences of cardinalities given by similar formulas. In each case, one can guess the Weights from the formula.

First there are some posets enumerated by the Fuss–Catalan numbers, namely the m-Tamari lattices introduced by F. Bergeron and L.-F. Préville-Ratelle [4] and also the simpler posets of m-Dyck paths under the relation of being weakly below. The larger family of rational Tamari lattices can also be considered, as they are counted by a similar formula.

Second, there are the Cambrian lattices associated with a finite Coxeter group W, all enumerated by the "Coxeter–Catalan number" for W. For simply-laced Coxeter groups, the derived equivalence between these posets, for a given W and all choices of Coxeter element, has been proved by Ladkani using quiver techniques in [37].

Third, there are the partial order on tilting modules (or positive clusters) for a Weyl group W, enumerated by the "positive Coxeter–Catalan numbers" for W. In this case too, for simply-laced Coxeter groups, the derived equivalence between these posets, for a given W and all choices of Coxeter element, has been proved by Ladkani using quiver techniques in [38].

5. Alternating Sign Matrices

Alternating sign matrices are combinatorial objects generalizing permutation matrices, that appeared in the Dodgson condensation algorithm for computing the determinant. The number of alternating sign matrices of size n is given by the famous formula

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}, \tag{5.1}$$

which was conjectured by Mills, Robbins and Rumsey [42], first proved by Zeilberger [58] and proved again by Kuperberg [35] using methods of statistical mechanics. It is an open problem to find an explicit bijection between alternating sign matrices and totally symmetric self-complementary plane partitions, which were enumerated by the same formula by Andrews in [1]. For a detailed account of the full story, see [7, 8].

Lemma 5.1. *The formula* (5.1) *is the Milnor number of the weak Weight*

$$(\{3k+2,\dots,n+k\}_{0\le k\le n-k-2}; 3n). \tag{5.2}$$

Proof. Formula (5.1) is the quotient of a product of factorials by a product of factorials. Consider the k-th and $(n-1-k)$-th factorials in the numerator, together with the k-th and $(n-1-k)$-th factorials in the denominator. This gives the quotient

$$\frac{(3k+1)!(3n-3k-2)!}{(n+k)!(2n-k-1)!}.$$

Assuming that $k < n-1-k$, this is

$$\frac{(2n-k)\cdots(3n-3k-2)}{(3k+2)\cdots(n+k)},$$

which can be written as the product

$$\prod_i \frac{D-d_i}{d_i},$$

where $(d_1,\dots,d_m) = (3k+2,\dots,n+k)$ and $D = 3n$. When n is odd and $2k = n-1$, the middle terms in the numerator and the denominator of (5.1) are both $(3k+1)!$ hence can be neglected. So the full expression is indeed associated with this weak Weight. □

In fact, this weak Weight should be a Weight, and its q-Milnor number should be a polynomial with positive integer coefficients. This can be checked for $n \le 30$, but there is no known combinatorial statistics to explain this property.

For small n, the Weights in this family are $\mathbb{A}_1, \mathbb{A}_2, \mathbb{E}_7, \mathbb{D}_7 \times \mathbb{E}_6$.

The expected Calabi–Yau dimension is then $(2\binom{n+1}{3}, 3n)$.

There are several natural partial orders on objects enumerated by formula (5.1). The first one is the enveloping lattice (or Dedekind–MacNeille completion) of the Bruhat order on the symmetric group $\mathscr{S}_n$, as proved in [39]. These posets do not meet the Coxeter criterion.

J. Striker has introduced in [53, §5], as part of a more general construction involving a choice among colors, two families of distributive lattices having (5.1) as cardinalities.

The first family (for the colors **b**lue, **y**ellow, **o**range and **g**reen in the terminology of [53]) has elements in bijection with the alternating sign matrices, and is in fact isomorphic to the enveloping lattice above.

The second family (for the colors **r**ed, **y**ellow, **o**range and **g**reen) has elements in bijection with the totally symmetric self-complementary plane partitions. Experimentally, the posets in this family do have the correct Coxeter polynomial for $n \leq 5$, hence satisfy the Coxeter criterion. So conjecturally, all these posets should be fractional Calabi–Yau.

As a side remark, one can note that the poset of size 42 in this family seems to be derived-equivalent to the posets of size 42 in the Catalan family, and they share the same Weight $\mathbb{D}_7 \times \mathbb{E}_6$.

Remark 5.2. *The same idea can be applied to other symmetry classes of plane partitions, which are often enumerated by a closed formula involving a product. This includes the full set of plane partitions inside an $a \times b \times c$ box and the famous formula of MacMahon. This also seems to work for totally symmetric plane partitions (A005157) and cyclically symmetric plane partitions (A006366). In the totally symmetric case, one observes amusing coincidences:*

- *in cardinality* 66*, for the Weight $\mathbb{A}_{11} \times \mathbb{E}_6$ with the Weight for the poset of tilting modules of type $\mathbb{F}_4$.*
- *in cardinality* 2431*, for the Weight $\mathbb{A}_{17} \times Z_{13} \times Q_{11}$ with the Weight for the poset of tilting modules of type $\mathbb{E}_7$.*

6. The West family

In his famous article [56], West introduced the notion of *2-stack sortable permutations* and conjectured that the number of such permutations on n letters is given by the formula

$$2\frac{(3n)!}{(2n+1)!(n+1)!}. \tag{6.1}$$

This was first proved by Zeilberger in [57].

The formula (6.1) can be written as

$$\frac{(3n)\cdots(2n+2)}{(3)\cdots(n+1)},$$

which comes from the weak Weight

$$(3,\ldots,n+1;3n+3). \tag{6.2}$$

Here again, it is not clear that the q-Milnor number is a polynomial in q with positive coefficients. One can check by computer that this is the case for $n \leq 50$. Assuming that this always holds and therefore that (6.2) defines a Weight, one can look for posets satisfying the Coxeter criterion.

For small n, the Weights in this family are $\mathbb{A}_1, \mathbb{A}_2, \mathbb{E}_6, \mathbb{A}_2 \times Z_{11}, \mathbb{E}_7 \times Z_{13}$. t The next Weight in this family can still be described using the notion of invertible polynomials, as considered in [5] and classified in [34]: it is product of $\mathbb{A}_2$ by the Weight corresponding to the chain type of parameter $(7,3,3,4)$.

In general, the Weights in the West family cannot be realized by invertible polynomials in the sense of Berglund–Hübsch. The first impossible case happens for $n = 8$, with Milnor number 9614.

Besides 2-stack sortable permutations, there are several other families of combinatorial objects with the same cardinality: left-ternary-trees [16], fighting fishes [18], non-separable planar maps [9] and synchronized Tamari intervals [46]. The last family is in bijection with the maximal cells in the diagonal of the associahedra [41, 15].

On these combinatorial objects, one can find several partial orders. One possibility is by restriction of partial orders on permutations (weak order, Bruhat order, *etc.*) to the subset of 2-stack sortable permutations. Another is to use the geometry of the diagonal of the associahedra, which is naturally oriented.

These tentatives have met no success so far, always failing the Coxeter criterion as soon as n is not very small. So, the question remains whether there does exist such a family of posets. One can even hope for the existence of posets whose Hasse diagrams would be the oriented 1-skeletons of a sequence of simple polytopes having their h-vectors given by A082680.

7. The Tamari-intervals family

In the 1960's, Tutte [55] has enumerated several kinds of rooted planar maps, obtaining elegant formulas. Among these, planar rooted triangulations are counted by the formula

$$2\frac{(4n+1)!}{(n+1)!(3n+2)!}. \tag{7.1}$$

This formula can be written as

$$\frac{(4n+1)\cdots(3n+3)}{(3)\cdots(n+1)},$$

hence comes from the weak Weight

$$(3,\dots,n+1;4n+4). \tag{7.2}$$

Once again, it is not clear if the q-Milnor number is a polynomial with positive coefficients. This property can be checked for $n \le 60$, but there is no known combinatorial statistics to explain this property.

For small n, the Weights in this family are $\mathbb{A}_1, \mathbb{A}_3, W_{13}, \mathbb{A}_4 \times W_{17}$.

Assuming that (7.2) always defines a Weight, one can look for posets satisfying the Coxeter criterion. Besides triangulations, there are now several other families of combinatorial objects counted by formula (7.1): the set of all intervals in Tamari lattices [10], extended fighting fishes [19], etc.

So far, no sequence of partial orders with the correct Coxeter polynomials has been found. The most natural partial order on Tamari intervals, involved in the relation with the diagonals of the associahedra, is defined by $[a,b] \le [a',b']$ if and only if $a \le a'$ and $b \le b'$. It does not meet the Coxeter criterion. The naive partial order by inclusion of intervals does not work either.

8. Green mutation poset for the cyclic quivers

In the theory of cluster algebras, one can associate mutation graphs to quivers. Using the notion of green mutations, one can define an orientation of the mutation graph. When the mutation graph is finite, one obtains finite posets, among which the Tamari lattice considered in Section 4. For details, see the survey [32].

Let us consider here the sequence of posets defined in this way, starting from the cyclic quivers on n vertices, with $n \ge 2$. In the classification by Fomin and Zelevinsky of quivers of finite type [23], these quivers have type $\mathbb{D}_n$. The number of elements in the posets (clusters) is therefore given by the Coxeter–Catalan number of type $\mathbb{D}_n$, which is

$$(3n-2)c_{n-1}, \tag{8.1}$$

where c_n is the Catalan number (4.1). This can be written as the Milnor number of the weak Weight

$$(4,2n,[6,9,12,\dots,3n-3];6n), \tag{8.2}$$

where the subsequence of degrees inside the bracket is an arithmetic progression of step 3.

One can check that this indeed defines Weights for $n \leq 50$.

For small $n \geq 2$, the Weights in this family are $\mathbb{A}_2 \times \mathbb{A}_2, \mathbb{A}_2 \times \mathbb{E}_7, \mathbb{A}_2 \times \mathbb{A}_5 \times \mathbb{D}_5, \mathbb{E}_{13} \times S_{1,0}$.

Experimentally, the green-mutation partial orders for the cyclic quivers satisfy the Coxeter criterion for the Weight given above. One therefore expects them to be fractional Calabi–Yau with the prescribed dimension.

Remark 8.1. *Let* Cat_n *be the Catalan Weight introduced in Section 4. One can note that the Weight associated above to the green mutation poset for the cyclic quiver of type* D_n *is a multiple of* Cat_{n-1}. *A similar phenomenon seems to happen for the sub-poset consisting of positive clusters, namely those not meeting the initial cluster, for the Weight* $\mathbb{A}_{n-1} \times \mathrm{Cat}_{n-1}$.

Appendix A. Derived categories of posets and Coxeter polynomials

Let $(P, \leq)$ be a finite partial order. One can define the incidence algebra of P over a field, and consider the category of finite dimensional modules over this algebra and its bounded derived category $\mathscr{D}(P)$. The category $\mathscr{D}(P)$ has finite global dimension and possesses Serre and Auslander–Reiten functors.

On the triangulated category $\mathscr{D}(P)$, the Auslander–Reiten translation functor τ is an auto-equivalence. It induces a linear map on the Grothendieck group $K_0(\mathscr{D}(P))$, which is a free abelian group of rank $|P|$. The matrix of this linear map in the basis made of classes of simple modules can be described as follows.

Pick any total order on P which is an extension of the partial order $\leq$. Let L_P be the triangular matrix with coefficient 1 in position (i,j) if $i \leq j$ and 0 elsewhere. The Coxeter matrix C_P is then $-L_P L_P^{-t}$, where L_P^{-t} is the transpose of the inverse of L_P. The Coxeter polynomial is the characteristic polynomial of the Coxeter matrix.

The Coxeter polynomial of a poset is concretely available in several computer algebra systems.

Appendix B. Tables and names

Here are small tables of named Weights, some of which have appeared in the article to describe the first few Weights in the families. The names come either from quivers, root systems or singularity theory.

For most of these Weights, one can find at least one poset whose derived category should be triangle-equivalent to the geometric category associated with the Weight.

Dynkin quivers	
$\mathbb{A}_n$	$(1; n+1)$
$\mathbb{D}_n$	$(2, n-2; 2n-2)$
$\mathbb{E}_6 = \mathbb{A}_2 \times \mathbb{A}_3$	$(3,4;12)$
$\mathbb{E}_7$	$(2,3;9)$
$\mathbb{E}_8 = \mathbb{A}_2 \times \mathbb{A}_4$	$(3,5;15)$

Elliptic root systems	
$\mathbb{E}_6^{(1,1)} = \mathbb{A}_2 \times \mathbb{A}_2 \times \mathbb{A}_2$	$(1,1,1;3)$
$\mathbb{E}_7^{(1,1)} = \mathbb{A}_3 \times \mathbb{A}_3$	$(1,1;4)$
$\mathbb{E}_8^{(1,1)} = \mathbb{A}_2 \times \mathbb{A}_5$	$(1,2;6)$

Arnold's unimodal singularities	
$E_{12} = \mathbb{A}_2 \times \mathbb{A}_6$	$(3,7;21)$
E_{13}	$(2,5;15)$
$E_{14} = \mathbb{A}_2 \times \mathbb{A}_7$	$(3,8;24)$
Z_{11}	$(3,4;15)$
Z_{12}	$(2,3;11)$
Z_{13}	$(3,5;18)$
$Q_{10} = \mathbb{A}_2 \times \mathbb{D}_5$	$(6,8,9;24)$
Q_{11}	$(4,6,7;18)$
$Q_{12} = \mathbb{A}_2 \times \mathbb{D}_6$	$(3,5,6;15)$
$W_{12} = \mathbb{A}_3 \times \mathbb{A}_4$	$(4,5;20)$
W_{13}	$(3,4;16)$
S_{11}	$(4,5,6;16)$
S_{12}	$(3,4,5;13)$
$U_{12} = \mathbb{A}_2 \times \mathbb{A}_2 \times \mathbb{A}_3$	$(3,4,4;12)$

Arnold's bimodal singularities	
$E_{18} = \mathbb{A}_2 \times \mathbb{A}_9$	$(3,10;30)$
E_{19}	$(2,7;21)$
$E_{20} = \mathbb{A}_2 \times \mathbb{A}_{10}$	$(3,11;33)$
Z_{17}	$(3,7;24)$
Z_{18}	$(2,5;17)$
Z_{19}	$(3,8;27)$
$Q_{16} = \mathbb{A}_2 \times \mathbb{D}_8$	$(3,7,9;21)$
Q_{17}	$(4,10,13;30)$
$Q_{18} = \mathbb{A}_2 \times \mathbb{D}_9$	$(6,16,21;48)$
W_{17}	$(3,5;20)$
$W_{18} = \mathbb{A}_3 \times \mathbb{A}_6$	$(4,7;28)$
S_{16}	$(3,5,7;17)$
S_{17}	$(4,7,10;24)$
$U_{16} = \mathbb{A}_2 \times \mathbb{A}_2 \times \mathbb{A}_4$	$(3,5,5;15)$

Quadrilateral singularities	
$J_{3,0} = \mathbb{A}_2 \times \mathbb{A}_8$	$(1,3;9)$
$Z_{1,0}$	$(1,2;7)$
$Q_{2,0} = \mathbb{A}_2 \times \mathbb{D}_7$	$(2,4,5;12)$
$W_{1,0} = \mathbb{A}_3 \times \mathbb{A}_5$	$(2,3;12)$
$S_{1,0}$	$(2,3,4;10)$
$U_{1,0} = \mathbb{A}_2 \times \mathbb{E}_7$	$(2,3,3;9)$

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