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The fractional CR curvature equation on the three-dimensional CR sphere

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(Recommended by Tobias Holck Colding)

Abstract. In this paper, we address the problem of prescribed fractional Q-curvature on a 3-dimensional sphere endowed with its standard CR structure. Since the associated variational problem is noncompact, we approach this issue using techniques of Bahri as the theory of critical points at infinity, using topological tools from generalizations of Morse theory. We prove some perturbative existence results.

1. Introduction

The CR Nirenberg problem in CR geometry is the well known Webster scalar curvature problem. Let (M,θ) be a compact $(2n+1)$-dimensional CR manifold equipped with a contact form θ. Given a prescribed function $K : M \to \mathbb{R}$, this problem amounts to solving the following curvature equation

$$\begin{cases} P_1^\theta u = K u^{\frac{n+2}{n}} \\ u > 0 \ \text{ on } M, \end{cases}$$

where $P_1^\theta = -\Delta_\theta + \frac{n}{2(n+1)} R_\theta$ is the CR invariant sublaplacian of M, Δ_θ the sublaplacian, and R_θ the Webster scalar curvature associated to θ. Introduced by Jerison and Lee in 1987 in [JL87], and better known as the CR Yamabe operator, P_1^θ was the first in a series of CR covariant operators that were discovered later. In 2005, Gover and Graham constructed and studied in [GG05] generalizations of the CR invariant sublaplacian which are the CR analogues of the conformally invariant powers of the Laplacian in conformal geometry. More recently in 2015, Frank, Gonzalez, Monticelli, and Tan introduced in [FGMT15] fractional powers of the sublaplacian on CR manifolds and constructed a class of fractional CR invariant operators on a CR manifold, from scattering theory on a Kähler-Einstein manifold [EMM91, GSB08, HPT08]. Denoted by P_s^θ these operators are actually pseudo-differential operators of order $2s$ where $s \in \mathbb{R}$, and whose principal symbol agrees with the pure fractional powers of the sublaplacian $(-\Delta_\theta)^s$. The main property of the operator P_s^θ is its CR covariance. Indeed, given a CR invariant change of contact form $\tilde{\theta} = u^{\frac{2}{n+1-s}}\theta$, the corresponding operator is given by the following transformation law:

$$P_s^{\tilde{\theta}}(f) = u^{-\frac{n+1+s}{n+1-s}} P_s^\theta(uf) \qquad \forall f \in \mathscr{C}^\infty(M).$$

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The formula for the Webster scalar curvature or the Paneitz–Branson Q-curvature can be extended, and the CR fractional Q-curvature for $\tilde{\theta}$ of order s can be defined as

$$Q_s^{\tilde{\theta}} = P_s^{\tilde{\theta}}(1),$$

which enjoys interesting covariant properties. Noting that for $f \equiv 1$ we have

$$P_s^{\theta}(u) = u^{\frac{n+1+s}{n+1-s}} P_s^{\tilde{\theta}}(1),$$

we can then formulate the CR fractional Nirenberg problem, or the CR fractional Q-curvature problem, as follows: find a CR invariant change of contact form $\tilde{\theta} = u^{\frac{2}{n+1-s}}\theta$ such that $Q_s^{\tilde{\theta}} = K$ where $K : M \to \mathbb{R}$ is a prescribed function. This problem reduces to solving the following curvature equation

$$\begin{cases} P_s^{\theta} u = K u^{\frac{n+1+s}{n+1-s}} \\ u > 0 \ \text{ on } M, \end{cases}$$

which is a nonlocal semilinear equation with critical power nonlinearity.

We are interested here in the case where M is the CR unit sphere $\mathbb{S}^{2n+1}$ with $n = 1$. Namely, let $\mathbb{S}^3$ be the unit sphere of $\mathbb{C}^2$ defined by

$$\mathbb{S}^3 = \left\{ \zeta = (\zeta^1, \zeta^2) \in \mathbb{C}^2 \,\middle|\, |\zeta|^2 = \sum_{j=1}^{2} |\zeta^j|^2 = 1 \right\}$$

endowed with its standard contact form

$$\theta = i(\overline{\partial} - \partial)|\zeta|^2 = i \sum_{j=1}^{2} \zeta^j d\overline{\zeta}^j - \overline{\zeta}^j d\zeta^j.$$

We wonder whether we can deform the contact form θ into a CR equivalent contact form $\tilde{\theta}$ whose associated fractional CR Q-curvature $Q_s^{\tilde{\theta}} = K$, where $K : \mathbb{S}^3 \to \mathbb{R}$ is a prescribed function, and $s \in (0,1)$. Thus, by posing $\tilde{\theta} = u^{\frac{2}{2-s}}\theta$, it is equivalent to finding a solution u of the following fractional Q-curvature equation:

$$\begin{cases} P_s^{\theta} u = K u^{\frac{2+s}{2-s}} \\ u > 0 \quad \text{on} \quad \mathbb{S}^3. \end{cases} \tag{1.1}$$

We recall that if the function K is constant, then problem (1.1) is the fractional CR Yamabe problem which was studied in [GMM18]. However, the fractional CR Q-curvature problem has been addressed in [CW17] and in [LW18]. Our aim is to handle such a question using some topological and dynamical tools related to the theory of critical points at infinity (see Bahri-Coron [BC85], Bahri [Bah89]) as well as to generalizations of Morse theory.

Let $K : \mathbb{S}^3 \longrightarrow \mathbb{R}$ be a $\mathscr{C}^2$ function. We assume that K satisfies the following nondegeneracy assumption:

($\mathbf{A_1}$) K has a finite set I of nondegenerate critical points, such that $\Delta_\theta K(y) \neq 0 \ \forall y \in \mathrm{I}$,

where Δ_θ is the usual sublaplacian operator of $\mathbb{S}^3$, (see e.g. [JL87]). We consider the following subset of critical points of K:

$$\mathrm{I}^+ = \left\{ y \in \mathrm{I} \,\middle|\, -\Delta_\theta K(y) > 0 \right\}, \tag{1.2}$$

and we introduce the following perturbative assumption:

($\mathbf{A_2}$) Assume that $K(\zeta) = 1 + \varepsilon K_0(\zeta)$, $\forall \zeta \in \mathbb{S}^3$, where $K_0 \in \mathscr{C}^2(\mathbb{S}^3)$ and $|\varepsilon|$ small.

Let $\operatorname{ind}(K,y)$ be the Morse index of K at its critical point y. For an integer k, we write $k \in \mathbf{N}$, if k satisfies the following condition: For any $y \in \mathrm{I}^+$, we have $3 - \operatorname{ind}(K,y) \neq k+1$. That is to say

$$\mathbf{N} = \left\{ k \in \mathbb{N} \,\middle|\, \forall\, y \in \mathrm{I}^+, 3 - \operatorname{ind}(K,y) \neq k+1 \right\}. \tag{1.3}$$

Our first existence result is:

Theorem 1.1. *Let $2/3 \leqslant s \leqslant 1$, and $K : \mathbb{S}^3 \to \mathbb{R}$ a $\mathscr{C}^2$ positive function satisfying* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$). *If*

$$\max_{k \in \mathbf{N}} \left| 1 - \sum_{\substack{y \in \mathrm{I}^+ \\ 3-\operatorname{ind}(K,y) \leqslant k}} (-1)^{3-\operatorname{ind}(K,y)} \right| \neq 0 \tag{1.4}$$

then for $|\varepsilon|$ small enough, there exists a solution for problem (1.1).

Observe that taking k to be 3, the condition $3 - \operatorname{ind}(K,y) \leq 3$ is obviously satisfied for all $y \in \mathrm{I}^+$, therefore we obtain the following corollary:

Corollary 1.1. *Let $2/3 \leqslant s \leqslant 1$, $K : \mathbb{S}^3 \to \mathbb{R}$ a $\mathscr{C}^2$ positive function satisfying* ($\mathbf{A_1}$) *and* ($\mathbf{A_2}$). *If*

$$\sum_{y \in \mathrm{I}^+} (-1)^{3-\operatorname{ind}(K,y)} \neq 1\,, \tag{1.5}$$

then for $|\varepsilon|$ small enough, there exists a solution for problem (1.1).

Comment. This Corollary recalls the main result of [LW18] which states that under condition ($\mathbf{A_1}$), if the sum

$$\sum_{y \in \mathrm{I}^+} (-1)^{3-\operatorname{ind}(K,y)} \neq 1\,,$$

then equation (1.1) has a solution. However, if in addition we impose the condition ($\mathbf{A_2}$), we will have a perturbative version of the result of [LW18], which coincides with Corollary 1.1. Therefore our Theorem 1.1 can be seen as a generalization of this perturbative version of the result of [LW18]. Note that an interpretation of the fact that the sum

$$\sum_{y \in \mathrm{I}^+} (-1)^{3-\operatorname{ind}(K,y)} \neq 1$$

is that the total topological contribution of all the critical points at infinity in the topology of the lower levels sets of J (see definition (2.3)) is nontrivial. A perturbative version of the result of [LW18] would state that under conditions ($\mathbf{A_1}$) and ($\mathbf{A_2}$), if the total topological contribution of all the critical points at infinity is nontrivial, there exists a solution to equation (1.1). But what if that total contribution is trivial? The main novelty of our Theorem 1.1 is to answer the following question: What can be said if the total contribution is trivial, but the partial contribution of a subset of critical points at infinity is nontrivial? Can we still take advantage of such partial topological information? Our Theorem 1.1 gives a sufficient condition to be able to deduce from such partial topological information the existence of solutions for (1.1). Contrary to the perturbative version of [LW18], our Theorem 1.1 works even when the sum

$$\sum_{y \in \mathrm{I}^+} (-1)^{3-\operatorname{ind}(K,y)} = 1\,,$$

but when this sum is different from 1, the result of [LW18] is more general than our Corollary 1.1, since it is true for any prescribed positive Morse function on $\mathbb{S}^3$, and not only for a positive Morse function close to a constant.

Note. Initially, the hypothesis imposed on the fractional order s is $2/3 \leqslant s < 1$. It is an unavoidable hypothesis in [LW18] since it appears in its main theorem. The inequality $2/3 \leqslant s$ is required in several places throughout [LW18], and since we use some results of [LW18] we must keep this lower bound of s. However, as we will see below in the proofs of our theorems, we could improve the upper bound by allowing the fractional order s to take the value 1.

In order to state our second perturbative existence result let us introduce for all integer $p \in \mathbb{N}$,

$$\mathrm{I}_p^+ = \left\{ y \in \mathrm{I}^+ ; \, \mathrm{ind}(K, y) = p \right\} \tag{1.6}$$

and let

$$\mathrm{I}_*^+ = \mathrm{I}^+ \setminus \mathrm{I}_0^+ . \tag{1.7}$$

We then have

Theorem 1.2. *Let $2/3 \leqslant s \leqslant 1$, $K : \mathbb{S}^3 \to \mathbb{R}$ a $\mathscr{C}^2$ positive function satisfying* $(\mathbf{A_1})$, $(\mathbf{A_2})$ *and the condition:*

$(\mathbf{A_3})$ $\exists\, \tilde{y} \in \mathrm{I}_1^+$, *such that,* $K(\tilde{y}) \geqslant K(z)$, $\forall z \in \mathrm{I}_2^+$.

Then provided that $|\varepsilon|$ *is small enough, there exists a solution for problem* (1.1).

2. Variational aspect, defect of compactness, and Palais-Smale sequences

We denote by H^s the completion of $\mathscr{C}^\infty(\mathbb{S}^3)$ with respect to the norm

$$\|u\|^2 = \int_{\mathbb{S}^3} P_s^\theta u \cdot u \, \theta \wedge d\theta$$

and recall that solutions of eq. (1.1) are the critical points in H^s of the energy functional

$$E(u) = \frac{1}{2}\|u\|^2 - \frac{2-s}{4} \int_{\mathbb{S}^3} K u^{\frac{4}{2-s}} \, \theta \wedge d\theta. \tag{2.1}$$

Let in the sequel $\Sigma = \left\{ u \in H^s \,\middle|\, \|u\| = 1 \right\}$ and $\Sigma^+ = \{ u \in \Sigma \mid u \geq 0 \}$. The Euler functional associated to problem (1.1) is denoted by J and is defined on H^s by:

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^3} K |u|^{\frac{4}{2-s}} \, \theta \wedge d\theta \right)^{\frac{1}{2}}} .$$

One knows that if v is a critical point of J in Σ^+, then $u = J(v)^{\frac{1}{s}} v$ is a solution for (1.1) in H^s, and hence the contact form $\tilde{\theta} = u^{\frac{2}{2-s}} \theta$ has its fractional Q-curvature $Q_s^{\tilde{\theta}} = K$.

Problem (1.1) is known to be delicate because $\frac{4}{2-s}$ is the critical exponent for the inclusion $H^s \hookrightarrow L^{\frac{4}{2-s}}$ which is continuous but not compact, and the functional J does not satisfy the Palais-Smale condition. In order to characterize the sequences that fail the Palais-Smale condition, we recall some definitions and notations. Let ω be the typical solution of the fractional CR Yamabe problem on the Heisenberg group $\mathbb{H}^1$ (see e.g. [LW18, GMM18]), defined for all $\xi = (z, t)$ in $\mathbb{H}^1$ by

$$\omega(\xi) = \frac{1}{|1 + |z|^2 - i t|^{2-s}} .$$

For each $(g, \lambda) \in \mathbb{H}^1 \times (0, \infty)$ we obtain another solution by left translation and dilation, $\omega_{(g,\lambda)}(\xi) = \lambda^{2-s} \omega\left(\lambda g^{-1} \xi\right)$. Now, for each $(a, \lambda) \in \mathbb{S}^3 \times (0, \infty)$, we introduce the solution of the fractional CR Yamabe problem on $\mathbb{S}^3$, denoted by $\delta_{(a,\lambda)}$ and defined by:

$$\delta_{(a,\lambda)}(\zeta) = \frac{1}{|1 + \zeta^2|^{2-s}} \, \omega_{(F(a),\lambda)} \circ F(\zeta) \tag{2.2}$$

for all $\zeta \in \mathbb{S}^3$, where F is a biholomorphic map from $\mathbb{S}^3 \smallsetminus \{-a\}$ onto $\mathbb{H}^1$, induced by the Cayley Transform (see [LW18, JL87, MU02]). Observe that one can choose a coordinate system such that point a coincides with the north pole of $\mathbb{S}^3$, and therefore, $F(a) = 0$.

Let us recall that a sequence (v_k) in $\sum^+$ is called a Palais-Smale (PS) sequence if $J'(v_k) \to 0$ and $J(v_k)$ is bounded. We associate with (v_k) the sequence $u_k = J(v_k)^{\frac{1}{s}} v_k$. It follows that $u_k \in H^s$. Since $J(u_k) = J(v_k)$, one can easily see that (v_k) is a (PS) sequence for J in $\sum^+$ if and only if (u_k) is a (PS) sequence for E in H^s–topology. (PS) sequences for the functional E have been completely identified by Theorem 1.1 in [GMM18] in the general case of $\mathbb{S}^{2n+1}$, and for the functional J by Lemma 5.2 in [LW18] in the special case of $\mathbb{S}^3$. If we assume that there are no solutions, then (PS) sequences for the functional J are linear combinations of bubble functions $\sum_{i=1}^p \delta_{(a_i,\lambda_i)}$, for some integer $p \geqslant 1$, such that concentration points a_i tend to distinct critical points y_i of K when speeds of concentration λ_i tend to infinity. Among these (PS) sequences, only sequences for which $y_i{}'s$ belong to I^+ give rise to a critical point at infinity, others are sometimes referred to as false critical points at infinity. Then, the limit of such a linear combination is called a critical point at infinity made out of p masses. From a dynamical point of view, it corresponds to a noncompact orbit for the flow of $-\nabla J$, the opposite of the gradient of J. We then have

Proposition 2.1. *Assume* (1.1) *has no solution. The only critical points at infinity of J in $\sum^+$ are combinations of p masses ($p \geqslant 1$), which are denoted by $\sum_{i=1}^p \delta_{(y_i,+\infty)} := (y_1,...,y_p)_\infty$, where the $y_i{}'s$ are distinct critical points of K in I^+.*

From a topological point of view, the property of a critical point at infinity which interests us is that, as a classical critical point, it induces a change of topology in the sets of the lower levels of the functional J, defined for $\beta \in \mathbb{R}$, by

$$J^\beta = \{u \in \textstyle\sum^+ \,\big|\, J(u) \leqslant \beta\}. \tag{2.3}$$

A critical point at infinity has a Morse index. In order to calculate it we need a Morse lemma and a Morse decomposition in some appropriate neighborhood of the critical point at infinity. To the best of our knowledge, this has been done only in the case of $\mathbb{S}^3$ in [LW18] (see Lemma 5.3). The unstable manifolds for K are the stable manifolds for $1/K$. It leads to the following result expressed with help of our notations:

Proposition 2.2. *The Morse index of a critical point at infinity made of p masses $(y_1,...,y_p)_\infty$ is the integer $\mu(y_1,...,y_p)_\infty = p-1+\sum_{i=1}^p 3-\mathrm{ind}(K,y_i)$.*

Notice that in Lemma 5.4 of [LW18], the critical point at infinity $(y_1,...,y_p)_\infty$ is identified to the p-uple $\tau_p = (y_1,...,y_p)$ where the $y_i{}'s$ are distinct critical points in I^+. The Morse index of τ_p is denoted there by $k(\tau_p) = 4p-1-\sum_{i=1}^p \mathrm{ind}(K,y_i)$ which is equal to $\mu(y_1,...,y_p)_\infty = p-1+\sum_{i=1}^p 3-\mathrm{ind}(K,y_i)$.

3. Proofs of the theorems

Proof of Theorem 1.1. First, consider the case where $2/3 \leqslant s < 1$. Since $K = 1+\varepsilon K_0$, the functional is

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^3}(1+\varepsilon K_0)|u|^{\frac{4}{2-s}}\theta\wedge d\theta\right)^{\frac{2-s}{2}}},$$

and for $\varepsilon = 0$, we obtain the fractional CR Yamabe functional

$$J_0(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{S}^3}|u|^{\frac{4}{2-s}}\theta\wedge d\theta\right)^{\frac{2-s}{2}}}$$

which possesses a 4-dimensional manifold of critical points

$$\mathcal{Z} = \{\delta_{(a,\lambda)} \,\big|\, (a,\lambda) \in \mathbb{S}^3 \times (0,\infty)\}.$$

Let σ be the lowest critical level of the functional J_0, that is to say: $\sigma = J_0(\delta_{(a,\lambda)}) = \min J_0$ on $\sum^+$. Since K_0 is bounded on $\mathbb{S}^3$, we derive that $J(u) = J_0(u)\big(1+O(\varepsilon)\big)$, where $O(\varepsilon)$ is independent of u and tends to 0 with ε. Hence, the next lemma:

Lemma 3.1. *Let $\eta > 0$, for $|\varepsilon|$ small enough, we have* $J^{\sigma+\eta} \subset J_0^{\sigma+2\eta} \subset J^{\sigma+3\eta}$.

On the other hand, the critical level of a critical point at infinity made of p masses $J\big((y_1,\dots,y_p)_\infty\big) = \sigma\big(\sum_{i=1}^{p} 1/K(y_i)^{\frac{2-s}{s}}\big)^{\frac{s}{2}}$ tends to $\sigma p^{\frac{s}{2}}$ as $\varepsilon \to 0$, since $K(y_i) = 1+\varepsilon K_0(y_i)$. Taking $\eta = \sigma/4$, we can assume $|\varepsilon|$ sufficiently small so that critical points at infinity made of two masses or more are above the level $\sigma+3\eta$, and those made of a single mass are below the level $\sigma+\eta$. Therefore, J has no critical points at infinity in the set

$$J_{\sigma+\eta}^{\sigma+3\eta} = \{u \in \textstyle\sum^+ \,\big|\, \sigma+\eta \leqslant J(u) \leqslant \sigma+3\eta\}.$$

Since, arguing by contradiction, we assume that (1.1) has no solution, it follows that $J^{\sigma+3\eta} \simeq J^{\sigma+\eta}$, where $\simeq$ denotes retracts by deformation. Using Lemma 3.1, we have that $J_0^{\sigma+2\eta} \simeq J^{\sigma+\eta}$. Now, we claim that $J^{\sigma+\eta}$ is a contractible set. Indeed, from what precedes, it is sufficient to prove that $J_0^{\sigma+2\eta}$ is a contractible set. Let $u_0 \in J_0^{\sigma+2\eta}$, and $t \mapsto u(t,u_0)$ the fractional CR Yamabe flow line. The flow verifies the following equation

$$\begin{cases} \frac{\partial u}{\partial t} = -\nabla J_0(u) \\ u(0) = u_0. \end{cases}$$

Using the results of [LW18], we know that the Palais-Smale condition is satisfied for the above equation for all $t > 0$, and when $t \to +\infty$, $u(t,u_0)$ converges to a single mass in $\mathcal{Z}$. Thus, $J_0^{\sigma+2\eta} \simeq \mathcal{Z}$. Since $\mathcal{Z}$ is a contractible set it follows that $J_0^{\sigma+2\eta}$ is a contractible set, and our claim follows. Now, let ℓ be the integer for which

$$\max_{k\in\mathbf{N}} \Big| 1 - \sum_{\{y\in \mathrm{I}^+ \,|\, \mu(y)_\infty \leqslant k\}} (-1)^{\mu(y)_\infty} \Big| \neq 0$$

is achieved. Here $\mu(y)_\infty = 3 - \mathrm{ind}(K,y)$ is the Morse index of the critical point at infinity of single mass $(y)_\infty$. We introduce the set

$$X_\ell^\infty = \bigcup_{\{y\in \mathrm{I}^+ \,|\, \mu(y)_\infty \leqslant \ell\}} \overline{W_u(y)_\infty}.$$

It is a stratified set of top dimension ℓ, and since it is made of unstable manifolds of critical points at infinity of a single mass, we derive from what precedes that $X_\ell^\infty \subset J^{\sigma+\eta}$. Observe that X_ℓ^∞ is contractible in $J^{\sigma+\eta}$, since $J^{\sigma+\eta}$ is a contractible set. More precisely, there exists a contraction $h : [0,1] \times X_\ell^\infty \to J^{\sigma+\eta}$, i.e. h continuous and such that $h(0,u) = u$ and $h(1,u) = \tilde{u}$ a fixed point in X_ℓ^∞. Let $H = h\big([0,1]\times X_\ell^\infty\big)$. H is a contractible stratified set of dimension $\ell+1$. Using the flow lines of $-\nabla J$, and the fact that $H \subset J^{\sigma+\eta}$, we have

$$H \simeq \bigcup_{\{y\in \mathrm{I}^+ \,|\, \mu(y)_\infty \leqslant \ell+1\}} W_u(y)_\infty.$$

Now, using the fact that $\ell \in \mathbf{N}$, there are no critical points at infinity of Morse index $\ell+1$. We derive that $H \simeq X_\ell^\infty$. Then, taking the Euler characteristic of both sides, we derive that

$$1 = \sum_{\{y\in \mathrm{I}^+ \,|\, \mu(y)_\infty \leqslant \ell\}} (-1)^{\mu(y)_\infty}.$$

This contradicts the assumption of Theorem 1.1. Therefore, Theorem 1.1 is proven in this case.

Now consider the case where $s = 1$. Then problem (1.1) is the classical problem of the prescribed Webster scalar curvature, and our Theorem 1.1 is reduced to the particular case $n = 1$ of Theorem 1.1 of [Yac11]. As this last theorem is true for $n \geq 1$, our Theorem 1.1 is true in this case, which completes the proof of Theorem 1.1.

Proof of Theorem 1.2. First consider the case where $2/3 \leqslant s < 1$. Arguing by contradiction, we assume that (1.1) has no solution. Let ∂ be the boundary operator in the sense of Floer-Milnor homology as introduced in [Yac02]. We recall that singular chains, in this homology, are generated by unstable manifolds of critical points of J, and, if $(y)_\infty$ is a critical point at infinity of Morse index $\mu(y)_\infty$, then

$$\partial\big(W_u(y)_\infty\big) = \sum_{\{(z)_\infty \mid \mu(z)_\infty = \mu(y)_\infty - 1\}} \mathrm{int}\big((y)_\infty, (z)_\infty\big) W_u(z)_\infty$$

where $\mathrm{int}\big((y)_\infty, (z)_\infty\big)$ is the intersection number of $W_u(y)_\infty$ and $W_s(z)_\infty$, the unstable (resp. stable) manifold of $(y)_\infty$ (resp. $(z)_\infty$), with respect of $-\nabla J$, (see [Bah96, Yac02]). Taking $y = \tilde{y}$ in I_1^+, given by assumption $(\mathbf{A_3})$ of Theorem 1.2, $W_u(\tilde{y})_\infty$ is a manifold of dimension $\mu(\tilde{y})_\infty = 2$, and satisfies $W_u(\tilde{y})_\infty \bigcap W_s(z)_\infty = \emptyset$ for any $(z)_\infty$ of Morse index 1, since under assumption $(\mathbf{A_3})$, $J\big((\tilde{y})_\infty\big) \leqslant J\big((z)_\infty\big)$ for all $z \in \mathrm{I}_2^+$. It follows that $\partial\big(W_u(\tilde{y})_\infty\big) = 0$, hence $W_u(\tilde{y})_\infty$ defines a cycle in $C_2(X^\infty)$ the group of 2-dimensional chains of

$$X^\infty = \bigcup_{y \in \mathrm{I}_*^+} W_u(y)_\infty.$$

Note that X^∞ is a stratified set of top dimension 2, since the highest Morse index of critical points at infinity $(y)_\infty$ where $y \in \mathrm{I}_*^+$, is less than or equal to 2. But then $W_u(\tilde{y})_\infty$ cannot belong to the boundary of a 2-dimensional chain of X^∞. Therefore $W_u(\tilde{y})_\infty$ defines a homological class of dimension 2 which is nontrivial in X^∞. Denoting the 2-dimensional homology group of X^∞ by $H_2(X^\infty)$, we then have

$$H_2(X^\infty) \neq 0 \tag{3.1}$$

Using the same arguments and notations of the proof of Theorem 1.1, we derive that X^∞ is contractible in $J^{\sigma+\eta}$ which retracts by deformation on X^∞. It follows that X^∞ is a contractible set, and therefore $H_k(X^\infty) = 0$ for all $k \geqslant 1$, which is in contradiction with (3.1). Theorem 1.2 is thereby proven in this case.
Now we consider the case where $s = 1$. Then problem (1.1) is the classical Webster scalar curvature problem, and our Theorem 1.2 becomes the particular case $n = 1$ of Theorem 1.3 of [Yac11] (or of the first part of Theorem 1.2 of [Yac13]). Since these two last theorems are true for $n \geq 1$, then Theorem 1.2 is true in this case too, which ends the proof of Theorem 1.2.

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