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Construction of Anosov flows in dimension 3 by gluing blocks

Neige Paulet

(Recommended by Boris Hasselblatt)

Abstract. We present a new result allowing us to construct Anosov flows in dimension 3 by gluing *building blocks*. By a building block, we mean a compact 3-manifold with boundary P, equipped with a $\mathscr{C}^1$ vector field X, such that the maximal invariant set $\bigcap_{t\in\mathbb{R}} X^t(P)$ is a saddle hyperbolic set, and such that ∂P is *quasi-transverse* to X, i.e., transverse except for a finite number of periodic orbits contained in ∂P. Our gluing theorem is a generalization of a recent result of F. Béguin, C. Bonatti, and B. Yu who only considered the case where ∂P is transverse to X. The quasi-transverse setting is much more natural. Indeed, our result can be seen as a counterpart of a theorem by Barbot and Fenley which roughly states that every 3-dimensional Anosov flow admits a canonical decomposition into building blocks (with quasi-transverse boundary). We will also present a number of applications of our theorem.

1. Anosov flows

A $\mathscr{C}^1$-flow X^t on a closed manifold $\mathscr{M}$ is said to be Anosov if there exists an X^t-invariant decomposition of the tangent bundle into the sum $T\mathscr{M} = E^{ss} \oplus \mathbb{R}.X \oplus E^{uu}$, where X is the vector field which generates the flow X^t, the vectors of E^{ss} are exponentially contracted and the vectors of E^{uu} exponentially expanded by the differential of the flow X^t in the future (see Figure 1).

This definition appeared in the 1960s, when D. Anosov studied the qualitative dynamical properties of the geodesic flow on Riemannian manifolds with negative curvature in his famous article [1], which became the prototype example of an Anosov flow.

As Anosov flows are structurally stable, one can hope to obtain a complete classification of their orbital equivalence classes on a given manifold by a finite number of combinatorial invariants. Such a classification in dimension 3 (the minimal dimension for an Anosov flow) is still today the motivation of many works and is far from being completed.

Anosov flows are also remarkable for the interactions which appear between the dynamics of the flow and the topology of the underlying manifold, and this particularly in dimension 3. For certain classes of 3-manifolds (torus bundles over the circle, or Seifert manifolds for example), the topology of the manifold almost completely determines the dynamics of the Anosov flow it can carry, up to orbital equivalence. On the other hand, there are 3-manifolds which carry several non-orbitally equivalent Anosov flows.

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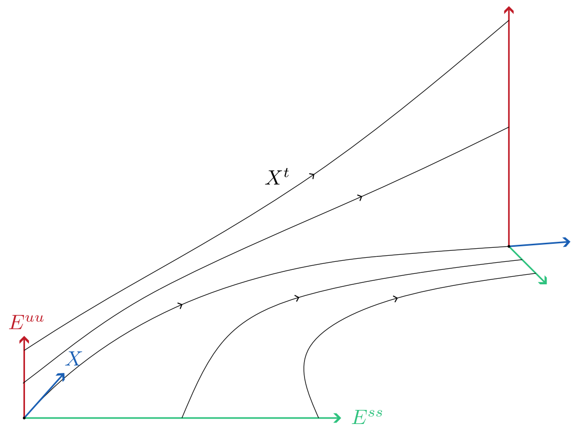

FIGURE 1. Local picture of an orbit of an Anosov flow

The main object of this paper is the construction of new examples of Anosov flows on 3-manifolds.

2. Previous construction of Anosov flows

The two standard examples of 3-dimensional Anosov flows are the following.

- **(geodesic flow)** Let (S,g) be a Riemannian surface. The *geodesic flow of* S is the flow $X^t\colon T^1S \to T^1S$ on the unitary tangent bundle of S, which maps the pair $(p,v)\in T^1S$ to the pair $X^t(p,v) = (c(t), c'(t))$ corresponding to the position and velocity at time t of the unique geodesic $c\colon \mathbb{R}\to S$ such that $(c(0),c'(0)) = (p,v)$. If *the Riemannian metric* g *is negatively curved*, then *the geodesic flow* X^t *on* $\mathcal{M} := T^1S$ *is Anosov.*
- **(suspension)** Let A be a matrix of $SL_2(\mathbb{Z})$ and $\bar{A}$ the automorphism induced on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The *suspension manifold of* A is the quotient $\mathcal{M}_A := \mathbb{T}^n \times [0,1]/(x,1) \sim (\bar{A}x, 0)$. It is a closed 3-manifold. The *suspension flow* X_A^t *of* A is the projection on $\mathcal{M}_A$ of the horizontal flow generated by the vector field $\partial/\partial t$ on the product $\mathbb{T}^n \times [0,1]$ equipped with the coordinates (x,t). If *the matrix* $A\in SL_2(\mathbb{Z})$ *is hyperbolic*, in other words it has no eigenvalues of modulus 1, then *the suspension flow of* A *is Anosov.*

These were the only examples of Anosov flows for decades. These two flows can also be described as the action of a one-parameter subgroup of a Lie group G, acting on a quotient of G by a co-compact lattice. Such a flow is called an *algebraic flow*. In the 1980s, the first examples of non-algebraic Anosov flows appeared. We present a number of constructions of such flows.

2.1. Franks-Williams construction. J. Franks and B. Williams construct in [18] the first non-transitive Anosov flow in dimension 3 by the *Blow up – Excise – Glue* technique. Let us give some details. Start with the suspension of the torus automorphism $A\colon (x,y)\mapsto (2x+y, x+y)$. This is an Anosov flow X_A^t on a closed 3-manifold $\mathcal{M}_A$. Let γ be the the periodic orbit induced by the fixed point $(0,0)$. One can perform a bifurcation (called a *bifurcation derived from Anosov*, see for example [19, Subsection 2.2.2] and Figure 2) on the flow in a neighborhood of γ to create an attractive periodic orbit γ^+. When removing a small tubular neighborhood of γ^+, one get a manifold P with a torus boundary, equipped with a vector field X transverse to ∂P, and such that the maximal invariant set $\bigcap_{t\in\mathbb{R}} X^t(P)$ is a hyperbolic repeller. By a result of Thurston, P is a hyperbolic manifold homeomorphic to the complement of the figure eight knot in $\mathbb{S}^3$. The pair (P, X) is an example of a building block. Consider another copy $P' = P$ equipped with the inverse vector field $X' = -X$. The construction consists in gluing the manifold P with a copy $P' = P$ equipped with the inverse vector field $X' = -X$ along their boundary. The authors shows that there is a way to glue this two components with a gluing map $\varphi : \partial P \to \partial P'$ such that the resulting vector field Z induced by X and X' on the closed 3-manifold $\mathcal{M} := P \cup P'/\varphi$ generates an Anosov flow. This flow admits two basic pieces which are the repeller of X in P and the attractor of X' in P', and therefore is not transitive. The key element is that the gluing map φ maps the unstable manifold of the repeller of X in P transverse to the stable manifold of the attractor of X' in P', which allows us to preserve the hyperbolic behaviour of the two pieces and create a global hyperbolic structure.

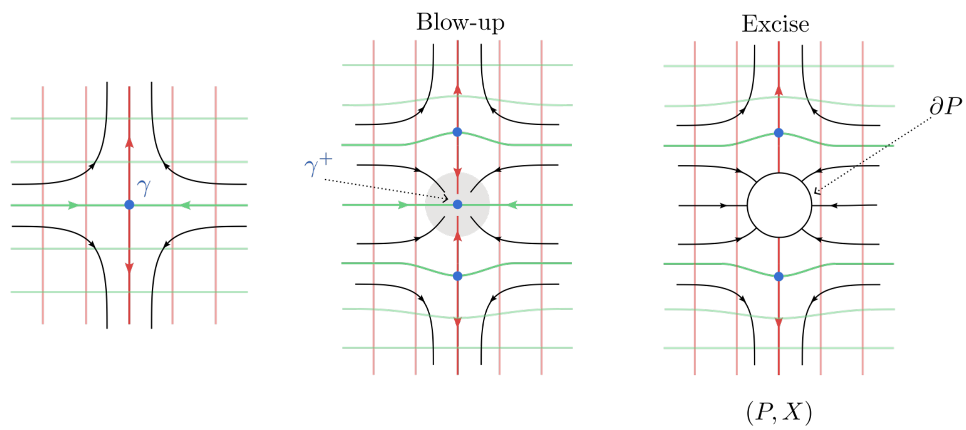

FIGURE 2. Attractive bifurcation derived from Anosov on the orbit γ

2.2. Handel–Thurston construction. M. Handel and W.P. Thurston construct in [20] the first non-algebraic transitive Anosov flow by performing a surgery on the geodesic flow of a hyperbolic surface. The setting is as follows. Consider a closed hyperbolic orientable surface Σ and a finite collection of disjoint closed simple geodesics $\{c_i\}$. Denote X^t the geodesic flow on the unit tangent bundle $\mathcal{M} = T^1\Sigma$. The union of the fibers above c_i in $\mathcal{M}$ is a torus T_i, which contains two periodic orbits of X^t, and transverse to X^t on the complementary of these orbits. It is said to be *quasi-transverse* to the flow (Figure 3). The Handel–Thurston surgery consists in cutting $\mathcal{M}$ along the T_i torus, and gluing back together the pieces P_j thus obtained by composing by Dehn twists $f_i \colon T_i \to T_i$ along the periodic orbits. Such gluing maps can be chosen so that they preserve the vector field while destroying the fibered structure. Under a certain positivity assumption on the twists f_i, the resulting flow is an Anosov flow Y^t on a manifold $\mathcal{N}$. This flow is always transitive because it preserves a volume form. The manifold $\mathcal{N}$ which carries this flow is a *graph manifold*, i.e., a collection of Seifert pieces which is not a Seifert space. By a result of Tomter [23] the resulting flow cannot be algebraic.

Both the example of Franks–Williams and Handel–Thurston are obtained by gluing *buildings blocks*, i.e., manifolds with boundary P equipped with a vector field X. Let us stress, however, one difference. In the Franks–Williams technique, the vector field is *transverse* to the boundary of the building blocks, whereas in Handel–Thurston the vector field is transverse to the boundary *except for a (nonzero) finite number of periodic orbits* contained in this boundary.

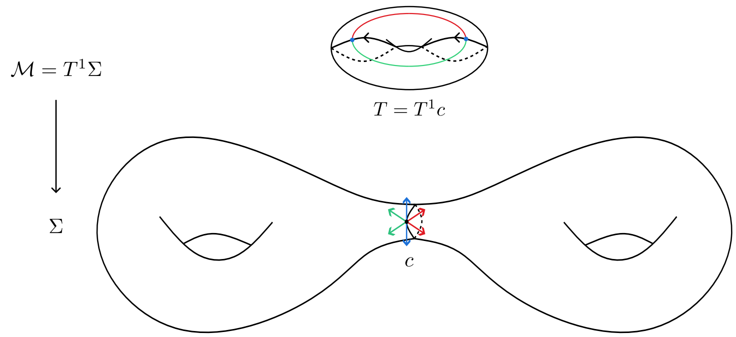

FIGURE 3. Fiber over a simple closed geodesic c

2.3. Bonatti–Langevin construction. The two previous examples are obtained by gluing the boundary components of building blocks that have been *cut out of a (blown-up) Anosov flow*. The example of C. Bonatti and R. Langevin [14] is the first example of an Anosov flow constructed from a building block which is not obtained by surgery on an initial standard Anosov flow. The block P is a circle bundle over a projective plane minus two disks, endowed with an explicit vector field X transverse to the boundary. The boundary of P consists of two tori, one T_1 along which the flow X^t enters P, and the other T_2 along which the flow exits P. The set of orbits of the flow included in the interior of P is reduced to a single hyperbolic saddle periodic orbit γ. In [14], the authors exhibit a gluing diffeomorphism $\varphi\colon T_2 \to T_1$ such that the flow induced by X on the quotient manifold $\mathscr{M} := P/\varphi$ is Anosov. This is the first example of transitive Anosov flow, transverse to an embedded torus, but not equivalent to a suspension. T. Barbot generalizes this construction in [4].

This technique is very different from the "surgery and gluing" techniques on standard Anosov flow of Franks–Williams and Handel–Thurston, where the gluing map is chosen so that is *does not destroy the hyperbolicity* from the initial (blown-up) Anosov flow that is already present everywhere. In the setting of Bonatti–Langevin, the dynamic inside the block is an explicit piece of a very simple Morse Smale flow, and the orbits that escape the block through the boundary do not have any hyperbolic behavior. The hyperbolicity along the new recurrent orbits is *created by the gluing process*.

2.4. Béguin–Bonatti–Yu construction. F. Béguin, C. Bonatti, and B. Yu have developed a general procedure for constructing Anosov flows by gluing "abstract building blocks" [11], in the spirit of the Bonatti–Langevin construction but where the manifold and the dynamics are not explicit. A *Béguin–Bonatti–Yu block* is a pair (P, X) where P is a compact manifold with boundary equipped with a $\mathscr{C}^1$ vector field X, transverse to ∂P, and such that the maximal invariant set $\bigcap_{t\in\mathbb{R}} X^t(P)$ forms a hyperbolic set whose strong stable and strong unstable bundles are of dimension 1. The authors show that under very general conditions, there is a way to glue the boundary components of P via a diffeomorphism $\varphi\colon \partial P \to \partial P$ which matches the components of the *exit boundary* (along which the flow exits P) with the components of the *entrance boundary* (along which the flow enters P) to obtain a closed manifold $P_\varphi := P/\varphi$ equipped with a vector field X_φ induced by X which is Anosov.

The gluing procedure of such blocks along their boundary is a powerful technique to show the flexibility of Anosov flows, and allows us to build Anosov dynamics on manifolds with a rich and complicated topology. It allows for example to construct a 3-manifold carrying a transitive and a non-transitive Anosov flow, and for every N, a manifold $\mathscr{M}_N$ carrying N pairwise non-orbitally equivalent Anosov flows.

However for Anosov flows constructed with this procedure, there always exist embedded tori which are transverse to the flow. The existence of such tori is not a common property. For example, the unit tangent bundle of a closed hyperbolic surface contain plenty of incompressible tori, but one can easily prove that none of these is isotopic to a torus transverse to the geodesic flow.

3. Decomposition of an Anosov flow: the modified JSJ decomposition

The counterpart of the Béguin–Bonatti–Yu gluing procedure is to find a good way to decompose a given Anosov vector field Z on a closed orientable 3-manifold $\mathscr{M}$ along tori into building blocks (P_i, X_i). Informally, we want the pieces to be as simple as possible (indecomposable), in a good position with respect to the vector field Z, and the decomposition to be unique. The Anosov vector field Z on $\mathscr{M}$ can then be reconstructed by gluing the blocks (P_i, X_i) along their boundary.

Recall the Jaco–Shalen–Johannson (JSJ) theorem which states that any irreducible connected orientable closed manifold $\mathscr{M}$ of dimension 3 can be cut along a minimal finite collection of incompressible tori $\mathscr{T} = \{T_1, \dots, T_n\}$ into pieces $P_1, \dots, P_m$ such that each P_i is either atoroidal or admits a Seifert fibration. The tori $T_1, \dots, T_n$ are unique up to isotopy (hence so are the 3-manifolds with boundary $P_1, \dots, P_m$). Any 3-manifold $\mathscr{M}$ that carries an Anosov vector field X is irreducible (since its universal covering is $\mathbb{R}^3$), and thus admits a JSJ-decomposition along incompressible tori. In [5], the authors study in detail the "optimal" position of an incompressible torus T embedded in an orientable manifold $\mathscr{M}$ with respect to the Anosov vector field X on $\mathscr{M}$. We will say that the torus T is *quasi-transverse* to X if T contains a finite (possibly zero) number of periodic orbits $\mathscr{O}_* = \{\mathscr{O}_1, \dots, \mathscr{O}_n\}$ of the flow, X is transverse to T on $T \smallsetminus \mathscr{O}_*$ is transverse to X on the complementary of the orbits $\mathscr{O}_*$, and the transverse orientation given by the vector field X on two adjacent components of $T \smallsetminus \mathscr{O}_*$ never coincide (see Figure 4).

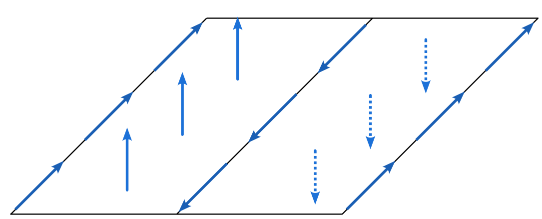

FIGURE 4. A torus T quasi-transverse to a vector field X, containing two periodic orbits of X

T. Barbot and S. Fenley show [5, Theorem 6.10] that any incompressible torus embedded in a 3-manifold $\mathscr{M}$ carrying an Anosov vector field X is homotopic to a torus quasi-transverse to X and *weakly embedded* (embedded outside the periodic orbits of X contained in the torus). This result gives a *modified JSJ decomposition* of an Anosov flow (see [10, Section 2.2] for a precise statement) which is unique up to homotopy along the flow. With the help of this decomposition of an Anosov flow in dimension 3, S. Fenley and T. Barbot have started the intensive study of Anosov flows on toroidal manifolds [5] and their classification in restriction to JSJ Seifert pieces and to some graph manifolds [6, 7, 8]. Let us stress that Anosov flows on atoroidal manifolds or in restriction to atoroidal pieces of the JSJ decomposition are still very poorly understood.

In conclusion, up to some technical details, any Anosov flow in a toroidal manifold can be canonically decomposed into blocks (P_i, X_i) where P_i is a manifold with boundary, and X_i is a $\mathscr{C}^1$ vector field on P_i which is quasi-transverse to the boundary ∂P_i. It is therefore natural to try to generalize the Béguin–Bonatti–Yu construction for such "quasi-transverse building blocks". This is the goal of our work.

4. Presentation of the Gluing Theorem

4.1. Definitions.

Definition 4.1 (Surface quasi-transverse to a vector field)**.** Let S be a transversally orientable surface embedded in a 3-dimensional manifold M equipped with a vector field X. We say that S is *quasi-transverse to the vector field X* if

1. S contains a finite collection $\mathscr{O}_* = \{\mathscr{O}_1, \dots, \mathscr{O}_n\}$ of periodic orbits of X
2. X is transverse to $S \smallsetminus \mathscr{O}_*$,
3. each orbit $\mathscr{O}_i \in \mathscr{O}_*$ is two-sided in S and the orbits of X cross these two local sides in two opposite directions.[1]

Note that a surface S transverse to the vector field X is a quasi-transverse surface with $\mathscr{O}_* = \emptyset$. We denote X^t the flow generated by the vector field X. Recall that a compact set Λ invariant by the flow of a vector field X on a manifold M is said to be *hyperbolic of index* $(1,1)$ for X if there exists an X^t-invariant decomposition of the tangent space of M over Λ into a sum $TM\big|_\Lambda = E^{ss} \oplus \mathbb{R}.X \oplus E^{uu}$ of 1-dimensional subbundles, and constants $\lambda > 1$ and $C > 0$ such that

$$\begin{aligned} \|(X^t)_* v\| &\geq C\lambda^t \|v\|, \quad \forall v \in E^{uu}, \quad \forall t \geq 0, \\ \|(X^t)_* v\| &\geq C\lambda^{-t} \|v\|, \quad \forall v \in E^{ss}, \quad \forall t \leq 0. \end{aligned}$$

for a Riemannian metric on M.

Definition 4.2 (Building block)**.** Let P be a compact 3-dimensional manifold with boundary, provided with a vector field X of class $\mathscr{C}^1$. We say that the pair (P, X) is a *building block* (or more simply a *block*) if

1. the boundary ∂P is quasi-transverse to the vector field X,
2. the maximal invariant set of the flow of X in P, denoted $\Lambda := \bigcap_{t\in\mathbb{R}} X^t(P)$, is an index $(1,1)$ hyperbolic set for the flow of X.

In the case where the collection $\mathscr{O}_*$ is empty, the boundary ∂P is transverse to the vector field X. We recover the definition of a *hyperbolic plug* in the sense of [11, Definition 3.1 and Definition 3.2].

Building blocks should be thought as the basic pieces of a building game, our goal being to build some Anosov flows by gluing a collection of such blocks together. From a formal point of view, a finite collection of building blocks can always be viewed as a single non-connected building block.

Consider such a building block (P, X). The orbits of $\mathscr{O}_*$ split ∂P into two open regions, transverse to the vector field X: one on which the field X is pointing inwards and one on which it is pointing outwards. We call them respectively the *entrance boundary* and *exit boundary*, and denote them P^{in} and P^{out} (Figure 6).

[1] Formally, this means that the transverse orientation induced by the vector field X on S on the two local sides of $\mathscr{O}_i$ cannot simultaneously coincide with a global transverse orientation of the surface S.

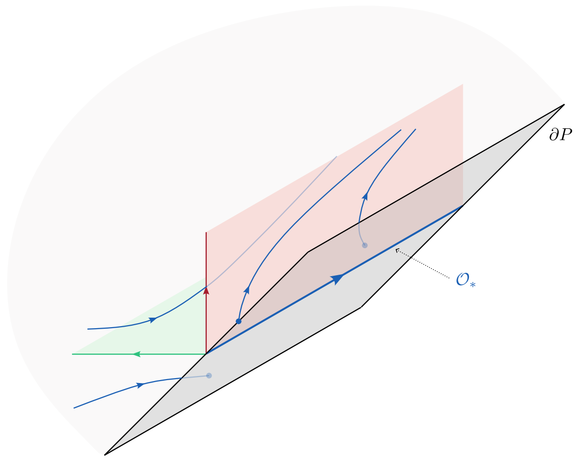

FIGURE 5. Boundary of a block (P, X) in the neighborhood of a periodic orbit $\mathscr{O}_i \in \mathscr{O}_*$

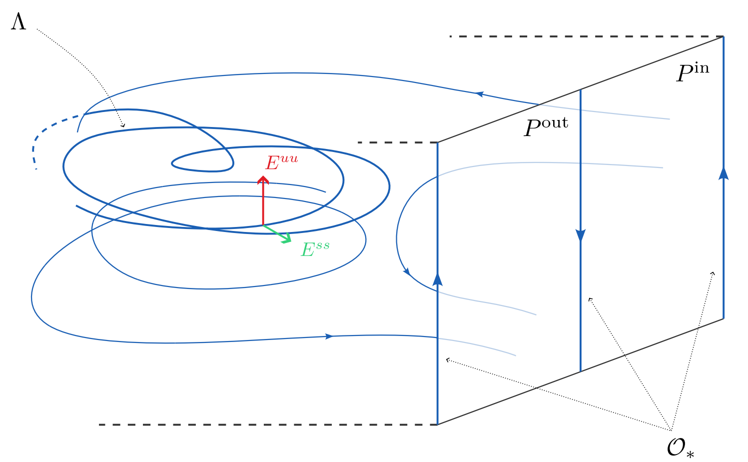

FIGURE 6. Building block (P, X) whose boundary contains two periodic orbits.

Definition 4.3 (Gluing map)**.** Let (P, X) be a building block. A *gluing map of* (P, X) is an involution $\varphi\colon \partial P \to \partial P$ which identifies pairwise the connected components of ∂P, maps P^{out} to P^{in}, and the oriented orbits of $\mathscr{O}_*$ to the oriented orbits of $\mathscr{O}_*$.

It is clear that these conditions are sufficient for the quotient space $P_\varphi := P/\varphi$ to be a closed smooth 3-dimensional manifold, and are necessary for X to induce a $\mathscr{C}^1$ vector field X_φ on P_φ. In the case where X induces a vector field X_φ on P_φ, there is in general no reason for the flow of X_φ to be an Anosov flow: the gluing map may create for instance an open set of periodic orbits for the flow of X_φ, which is an obvious obstruction. Necessary conditions to obtain an Anosov flow are encoded in the stable and unstable manifolds of the block and the way they are glued.

Let $\mathscr{W}^s$ and $\mathscr{W}^u$ be respectively the stable and the unstable set of the maximal invariant set $\Lambda := \bigcap_{t\in\mathbb{R}} X^t(P)$ of the block (P, X). The hyperbolicity of index $(1,1)$ of Λ implies that $\mathscr{W}^s$ and $\mathscr{W}^u$ are 2-dimensional laminations, embedded in P, transverse to each other.

Proposition and Definition (Boundary lamination)**.** The set $\mathscr{L} := (\mathscr{W}^u \cup \mathscr{W}^s) \cap \partial P$ is a 1-dimensional lamination on ∂P, called the *boundary lamination of* (P, X) (see Figure 7).

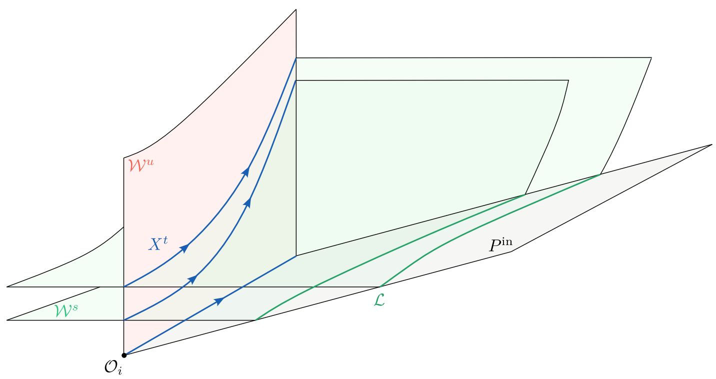

FIGURE 7. Boundary lamination $\mathscr{L}$ on the entrance boundary P^{in} in the neighborhood of a periodic orbit $\mathscr{O}_i \in \mathscr{O}_*$

We define the following conditions on the boundary lamination and the gluing map.

Definition 4.4.

(1) We say that the boundary lamination $\mathscr{L}$ is a *pre-foliation* if it can be extended by a foliation on ∂P.
(2) We say that the boundary lamination $\mathscr{L}$ is *filling*, or that the block (P, X) is *full*, if $\mathscr{L}$ is a pre-foliation and no connected component of the complementary $\partial P \smallsetminus \mathscr{L}$ is bounded by compact leaves.
(3) We say that a gluing map φ of (P, X) (see Definition 4.3) is *strongly quasi-transverse* if the pair of laminations $(\varphi_*(\mathscr{L} \smallsetminus \mathscr{O}_*), \mathscr{L} \smallsetminus \mathscr{O}_*)$ on $\partial P \smallsetminus \mathscr{O}_*$ can be extended by a pair of transverse foliation on $\partial P \smallsetminus \mathscr{O}_*$.

If $\mathscr{L}$ and φ satisfies the 3 items of Definition 4.4, it follows that the connected components of ∂P are tori or Klein bottle and the connected components of $\partial P \smallsetminus (\varphi_* \mathscr{L} \cup \mathscr{L})$ are rectangles bounded by two disjoint arcs of leaves of $\mathscr{L}$ and two disjoint arcs of leaves of $\varphi_* \mathscr{L}$. The condition 1 and 3 are necessary conditions to glue the boundary components of (P, X) together to create an Anosov vector field X_φ on P_φ. Indeed the manifold P_φ is then provided with a pair of invariant transverse foliations which are the stable and unstable foliation $\mathscr{F}^s$ and $\mathscr{F}^u$ of the Anosov flow. The surface ∂P projects onto P_φ as a surface S quasi-transverse to X_φ, and the pair $(\mathscr{F}^s, \mathscr{F}^u)$ induces a pair of 1-dimensional foliations on S which coincide along the periodic orbits of X_φ contained in S, and are transverse to each other on the complementary of these periodic orbits, and which contains the projection of the pair $(\mathscr{L}, \varphi_*(\mathscr{L}))$. The condition 2 is a technical condition that should not be necessary, but our proof relies heavily on this assumption.

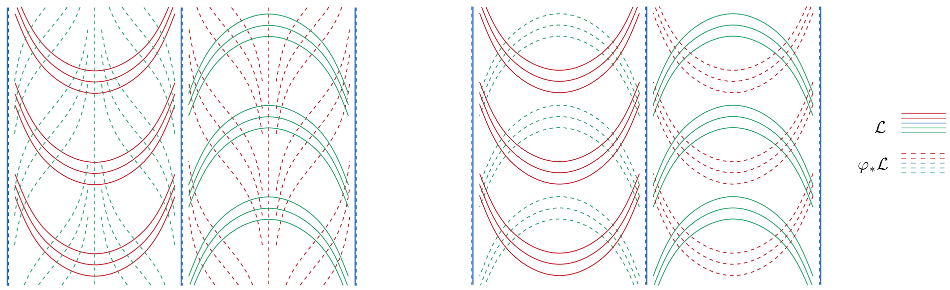

FIGURE 8. An example of a strongly quasi-transverse gluing diffeomorphism φ on the left, and quasi-transverse but not strongly quasi-transverse on the right

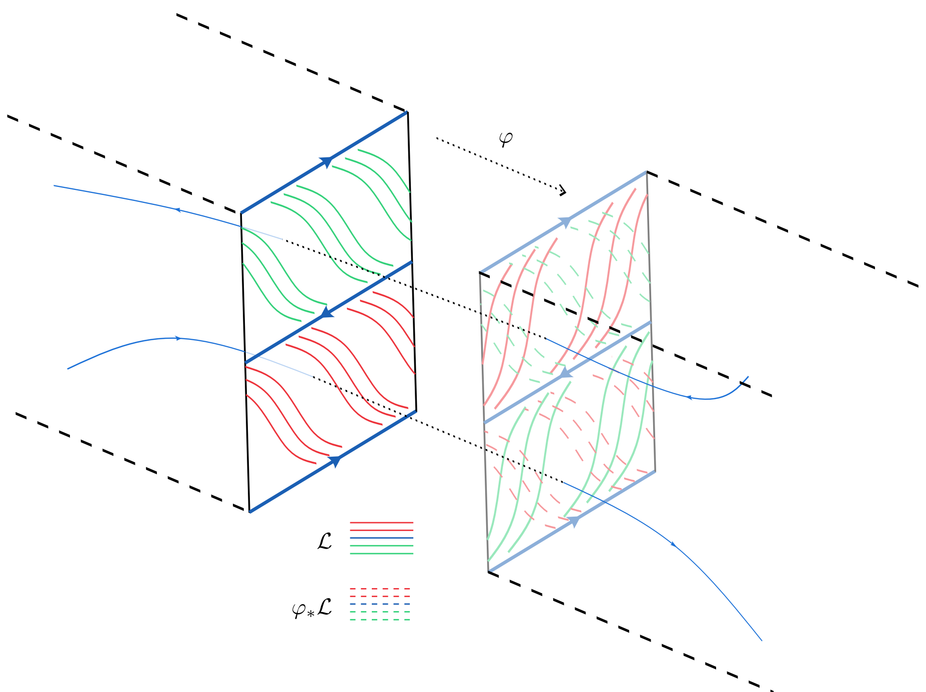

FIGURE 9. Building block (P, X) with filling boundary lamination $\mathscr{L}$, and a strongly quasi-transverse gluing diffeomorphism φ

In order to do our construction, we want to reduce the block (P, X) to a "normalized" form by an isotopy among the building blocks. For a strongly quasi-transverse gluing map φ, we want to modify it by isotopy among gluing maps while preserving the pattern of the intersection of the lamination $\varphi_*\mathscr{L}$ with $\mathscr{L}$ on ∂P. Let us define formally the equivalence relation that we obtain.

Definition 4.5 (Strongly isotopic triples)**.** Let (P_0, X_0, φ_0) and (P_1, X_1, φ_1) be two building blocks provided with gluing maps. The triples are said to be *strongly isotopic* if

(1) there exists a continuous family (P_t, X_t, φ_t) of building blocks endowed with gluing map,[2]
(2) there exists a continuous family of homeomorphisms $h_t\colon \partial P_0 \smallsetminus \mathscr{O}_{0,*} \to \partial P_t \smallsetminus \mathscr{O}_{t,*}$ which preserve the boundary lamination, such that $h_0 = \mathrm{Id}$ and h_1 maps the pair $(\mathscr{L}_0 \smallsetminus \mathscr{O}_{0,*}, (\varphi_0)_*(\mathscr{L}_0 \smallsetminus \mathscr{O}_{0,*}))$ on the pair $(\mathscr{L}_1 \smallsetminus \mathscr{O}_{1,*}, (\varphi_1)_*(\mathscr{L}_1 \smallsetminus \mathscr{O}_{1,*}))$.

Unlike the Béguin–Bonatti–Yu blocks, one cannot simply ask for a continuous family (P_t, X_t, φ_t) of blocks and strongly quasi-transverse gluing maps. The presence of periodic orbits contained in the boundary makes this isotopy a relation too rigid for what we want to do. Our strong isotopy relation above is a slightly weakened version of the *strong isotopy relation of Béguin–Bonatti–Yu blocks and gluing maps* [11] in the case where the collection $\mathscr{O}_*$ of periodic orbits contained in the boundary of the block is empty, due to the poor regularity of the h_t family (assumed to be continuous and not $\mathscr{C}^1$). This strong triple isotopy relation is nevertheless sufficient, as it guarantees that

- the quotient manifolds P_0/φ_0 and P_1/φ_1 are homeomorphic;
- the pattern of the bi-laminations $(\mathscr{L}_0, (\varphi_0)_*\mathscr{L}_0)$ and $(\mathscr{L}_1, (\varphi_1)_*\mathscr{L}_1)$ are the same;
- there are continuations[3] of the blocks (P_0, X_0) and (P_1, X_1) which are orbitally equivalent.

The pattern of the intersection of the lamination $\varphi_*\mathscr{L}$ with $\mathscr{L}$ is a crucial data for the analysis of the dynamic of the new vector field X_φ. It is a key element for constructing non-orbitally equivalent Anosov vector fields X_i on a closed manifold $\mathscr{M}$, by gluing the boundary components of a block (P, X) by isotopic gluing maps φ_i, but whose transverse intersection patterns of the lamination $\mathscr{L}$ and $\varphi_*\mathscr{L}$ are *not equivalent* [11, 16]. Conversely, it is also the key element of the *uniqueness theorem* of an Anosov flow obtained by gluing a Béguin–Bonatti–Yu block. In [12], F. Béguin and B. Yu show that if Z_0 and Z_1 are two transitive Anosov vector fields, respectively obtained by gluing a block (P_0, X_0) and (P_1, X_1) via a diffeomorphism φ_0 and φ_1, and such that the associated triples are strongly isotopic, then the fields Z_0 and Z_1 are orbitally equivalent. This question is not treated here, but it seems natural that this uniqueness result carries over to the case where the boundary of the blocks is quasi-transverse to the vector field. More precisely, we make the following conjecture:

Conjecture 4.6. *Let (P_0, X_0, φ_0) and (P_1, X_1, φ_1) be two building blocks equipped with gluing map such that the triples are strongly isotopic. For $i = 0, 1$, we assume that X_i induces a $\mathscr{C}^1$ vector field Z_i on the closed manifold $\mathscr{M}_i := P_i/\varphi_i$, such that $\mathscr{M}_i$ is orientable and Z_i is a transitive Anosov vector field. Then Z_0 on $\mathscr{M}_0$ and Z_1 on $\mathscr{M}_1$ are orbitally equivalent.*

4.2. Gluing Theorem. The proof of the following gluing theorem can be found in [22, Theorème 1], and is the goal of the forthcoming paper [21].

Theorem 1 (Gluing theorem)**.** *Let (P, X) be a full building block, and φ be a strongly quasi-transverse gluing map of (P, X). There exists a triple (P_1, X_1, φ_1) strongly isotopic to (P, X, φ) such that X_1 induces an Anosov vector field on the closed 3-manifold $P_{\varphi_1} := P_1/\varphi_1$.*

[2] More formally, we ask for the manifolds P_t to be embedded in a common 3-manifold $\tilde{P}$ equipped with extensions $\tilde{X}^t$ of X_t and the family to be continuous in the $\mathscr{C}^1$-topology.

[3] That is, an embedding of the block in the interior of a 3-manifold endowed with an extension of the vector field.

This theorem is an analog of the Béguin–Bonatti–Yu gluing theorem [11, Theorem 1.5] in the case where the set $\mathcal{O}_*$ of periodic orbits of X contained in ∂P is empty, and extends it to blocks containing attractors or repellers. Moreover it includes the Franks–Williams surgery, the Handel–Thurston surgeries and their generalizations. Finally, it allows us to consider the most natural building blocks for Anosov flows in dimension 3, in the sense that the quasi-transverse position is the "optimal" position of an incompressible closed surface embedded in an Anosov flow according to the work of T. Barbot and S. Fenley mentioned above.

4.3. Some ideas of the proof. The proof of our gluing theorem is a generalization of that of Béguin–Bonatti–Yu's. Although it uses the same ideas, the generalization is far from being straightforward, and is quite technical. It consists of the following 5 steps.

(1) We modify a candidate triple (P, X, φ) by strong isotopy to put it in a "normalized form". Informally, a triple (P, X, φ) is said to be *normalized* if
   - the dynamic of X is linear in a neighborhood of the maximal invariant Λ,
   - the multipliers of all the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{O}_*$ contained in the boundary are the same,
   - there is a pair of transverse invariant 2-dimensional foliations $(\mathcal{G}^s, \mathcal{G}^u)$ on P that extends the laminations $(\mathcal{W}^s, \mathcal{W}^u)$, and which are affine in the neighborhood of the Λ,
   - the boundary ∂P is in a canonical position in a neighborhood of the periodic orbits $\mathcal{O}_i \in \mathcal{O}_*$,
   - the gluing map φ is trivial in a neighborhood of the orbits $\mathcal{O}_*$.

   For the rest of the proof, we work with a normalized triple.

(2) In step 2, we study the hyperbolicity properties of the crossing map $f_{\text{out,in}}\colon P^{\text{in}} \to P^{\text{out}}$ of the flow of X from the entrance boundary P^{in} to the exit boundary P^{out}. We show that $f_{\text{out,in}}$ expands the direction tangent to $\mathcal{G}^u \cap P^{\text{in}}$ arbitrarily strongly in a small neighborhood of the boundary lamination $\mathcal{L}$, and the analogous result for the inverse map along the direction tangent to $\mathcal{G}^s$.

(3) In step 3, we show how to "spread" the natural hyperbolicity of $f_{\text{out,in}}$ by a coordinate change on the boundary of P. We want the crossing map to expands the direction tangent to $\mathcal{G}^u$, not only in the neighborhood of $\mathcal{L}$, but almost on all P^{in} (and the analogous for the inverse map along the direction tangent to $\mathcal{G}^s$). This spreading will depends on 3 parameters.

(4) In step 4, we show that we can use this change of coordinates to modify the gluing map φ turning it into a gluing map ψ strongly isotopic to φ, depending on the 3 parameters. The goal is to create hyperbolicity along the new recurrent orbits of the induced flow X_ψ, in a way which is compatible with the natural hyperbolicity of the initial flow X.

(5) In the final step 5, we show that for such a choice of parameters and gluing map ψ, the flow X_ψ induced by the initial flow X on the closed manifold P_ψ admits a hyperbolic structure along all orbits, and therefore is Anosov, which completes the proof of the theorem.

Actually, we did a small cheat. The latter strategy can be carried out under the additional assumption that the maximal invariant set of the block contains no attractor nor repeller, which is actually the statement of [22, Theorème 1]. A trick allows us to reduce the general case to this situation.

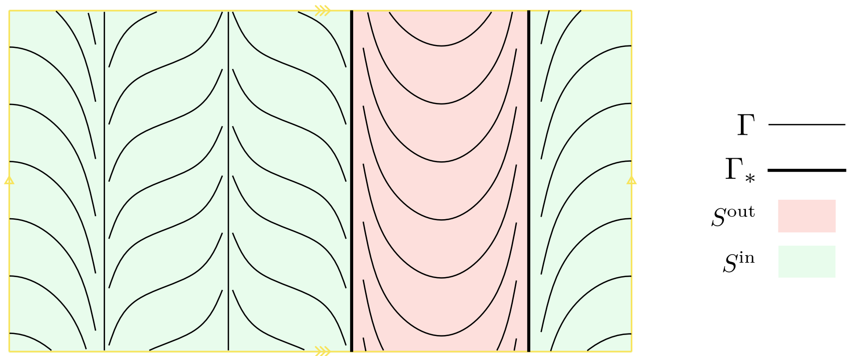

FIGURE 10. A quasi-Morse–Smale foliation on $S = \mathbb{T}^2$

4.4. Transitivity criterion. We show a transitivity criterion of an Anosov vector field X_φ on a closed manifold P_φ obtained by gluing a full block (P, X) with a strongly quasi-transverse gluing map φ. We define an oriented graph $G = G(P, X, \varphi)$ analogous to the Smale's graph, associated to a building block (P, X) and to a gluing map φ, as follows:

- the vertices are the basic pieces Λ_i of the hyperbolic maximal invariant set Λ of (P, X),
- there exists an oriented edge from Λ_i to Λ_j if and only if $W^u(\Lambda_i)$ intersects $W^s(\Lambda_j)$, or $\varphi(W^u(\Lambda_i))$ intersects $W^s(\Lambda_j)$.

Proposition 4.7. *the Anosov vector field X_φ on P_φ is transitive if and only if the graph $G(P, X, \varphi)$ is strongly connected.*

We refer to [22, Proposition 6.2] for the proof.

5. Applications

We now present some applications of our theorem. The proofs can be found in [22].

5.1. Realization of a quasi-transverse bi-foliation in an Anosov flow. As first application, we prove that any type of *quasi-transverse bi-foliation* on a torus can be realized as the trace of the stable and unstable foliation on a quasi-transverse torus embedded in a transitive Anosov flow. More formally,

Definition 5.1 (Quasi-Morse–Smale foliation)**.** Let $\mathscr{F}$ be a 1-dimensional foliation on a closed orientable surface S. We say that $\mathscr{F}$ is a *quasi-Morse–Smale* foliation if it satisfies the following conditions:

(1) There exists a finite number of compact leaves $\Gamma = \{\gamma_1, \ldots, \gamma_N\}$;
(2) Each half non-compact leaf accumulates on a single compact leaf;
(3) Each oriented compact leaf γ has a contracting or expanding holonomy on each of its two side.

The (possibly empty) set of elements of Γ such that the holonomy is contracting on one side and expanding on the other is denoted by Γ_*, and called the set of *marked compact leaves.* We further require

(4) There exists a splitting $S \setminus \Gamma_* = S^{\text{in}} \cup S^{\text{out}}$ into two disjoint open sets such that each leaf $\gamma_* \in \Gamma_*$ is adjacent to a connected component of S^{in} and to a connected component of S^{out}.

Definition 5.2 (Quasi-transverse bifoliation)**.** Let $(\mathscr{F}_1, \mathscr{F}_2)$ be a pair of quasi-Morse–Smale foliations on an orientable surface S. We say that $(\mathscr{F}_1, \mathscr{F}_2)$ is a *quasi-transverse bifoliation* if

(1) $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same set of marked leaves $\Gamma_{*,1} = \Gamma_{*,2} = \Gamma_*$,
(2) $\mathscr{F}_1$ and $\mathscr{F}_2$ are transverse on the complementary $S \setminus \Gamma_*$,
(3) $S_1^{\text{in}} = S_2^{\text{out}}$,

where $S \setminus \Gamma_{*,i} = S_i^{\text{in}} \cup S_i^{\text{out}}$ is the decomposition of $S \setminus \Gamma_{*,i}$ relative to $\mathscr{F}_i$.

We can provide the compact leaves of a quasi-Morse–Smale foliation $\mathscr{F}$ on a surface S with a canonical orientation: this is the orientation such that the holonomy of the oriented compact leaf is contracting on S^{in} and expanding on S^{out}. We say that $\mathscr{F}$ is *dynamically oriented.* For a pair $(\mathscr{F}_1, \mathscr{F}_2)$ of dynamically oriented quasi-transverse bifoliation on an *oriented* torus $S = \mathbb{T}^2$, we associate a finite combinatorial data $\sigma = \sigma(\mathscr{F}_1, \mathscr{F}_2)$ in the following way.

Definition 5.3 (Combinatorial type of quasi-transverse bi-foliation)**.** Let

$$\sigma = \sigma(\mathscr{F}_1, \mathscr{F}_2) \colon \mathbb{Z}/n\mathbb{Z} \longrightarrow \{1,2\} \times (\{\rightarrow, \leftarrow\} \times \{\uparrow, \downarrow\} \times \{\rightarrow, \leftarrow\})$$

be the map defined by

(1) $\sigma(k) = (1, *)$ if and only if γ_k is a leaf of $\mathscr{F}_1$, and $\sigma(k) = (2, *)$ if and only if γ_k is a leaf of $\mathscr{F}_2 \setminus \Gamma_*$;
(2) $\sigma(k) = (*, (*, \uparrow, *))$ if and only if the leaf γ_i is freely homotopic to γ_0 as oriented paths;
(3) $\sigma(k) = (i, (\rightarrow, *, *))$ if and only if the $\mathscr{F}_i$-holonomy of γ_k on its left-hand side [4] is contracting;
(4) $\sigma(k) = (i, (*, *, \leftarrow))$ if and only if the $\mathscr{F}_i$-holonomy of γ_k on its right-hand side is contracting.

We say that σ is a *combinatorial type of the quasi-transverse bi-foliation* $(\mathscr{F}_1, \mathscr{F}_2)$.

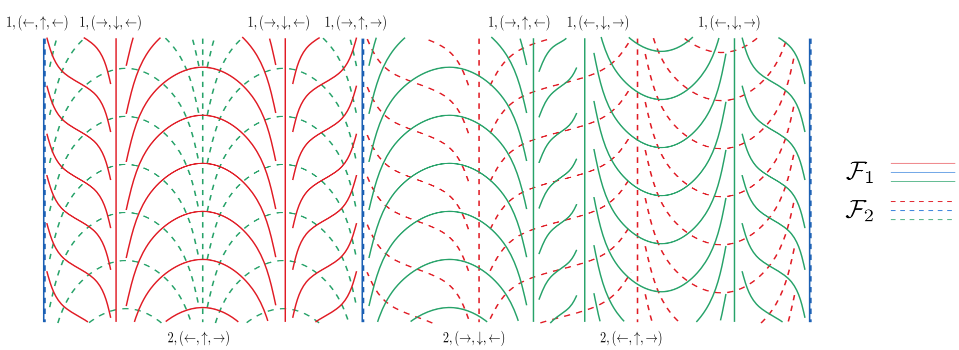

FIGURE 11. An example of a (complicated) quasi-transverse bi-foliation on the torus and combinatorial type

Proposition 5.4 ([22, Proposition 7.5.4])**.** *Let σ be a combinatorial type of a quasi-transverse bi-foliation. There exists a transitive Anosov vector field Z on an oriented 3-manifold $\mathscr{M}$ and an incompressible torus T embedded in $\mathscr{M}$, quasi-transverse to Z, such that the trace of the stable and unstable foliation $\mathscr{F}^s$ and $\mathscr{F}^u$ on T induces a bi-foliation $(\mathscr{F}_1, \mathscr{F}_2)$ on T of combinatorial type σ.*

[4] the orientation of the torus S and the first leaf γ_0 determines which side of γ_i is its left-hand and right-hand side

The proof consist in constructing a full transitive connected orientable building block (P, X) with two boundary components T_1 and T_2, and a prescribed "type" of quasi-Morse–Smale lamination as boundary lamination, corresponding to the type of $\mathscr{F}_i$ on T_i. We perform a construction that allow us to directly use the result of [11, Theorem 1.10] on attracting blocks of Béguin–Bonatti–Yu with prescribed boundary foliation. We can glue the boundary components of (P, X) with a strongly quasi-transverse gluing map $\varphi : T_1 \to T_2$ respecting the pattern of the given combinatorial type $\sigma(\mathscr{F}_1, \mathscr{F}_2)$. We can then use Theorem 1, and get an Anosov flow satisfying Proposition 5.4.

5.2. Embedding of a building block into an Anosov flow. As a further application of Theorem 1, we show that any full orientable block can be embedded in an Anosov flow:

Proposition 5.5 ([22, Proposition 8.0.1])**.** *For any (transitive) full orientable block (P, X), there exists a (transitive) Anosov vector field Z on a closed orientable 3-manifold $\mathscr{M}$, such that (P, X) is embedded in $(\mathscr{M}, Z)$. More precisely, there exists a finite collection of incompressible tori $\mathscr{T}$ embedded in $\mathscr{M}$, quasi-transverse to Z, such that the closure of one connected component of $\mathscr{M} \smallsetminus \mathscr{T}$ is a compact submanifold with boundary diffeomorphic to P and such that the restriction of Z on P is orbitally equivalent to X.*

The idea is to construct a transitive block (Q, Y) whose boundary lamination $\mathscr{L}_Y$ matches a lamination strongly quasi-transverse to the boundary lamination $\mathscr{L}_X$ of (P, X), and to glue the boundaries of the two blocks via a strongly quasi-transverse gluing map $\varphi : \partial Q \to \partial P$. We can then apply Theorem 1 to say that, up to strong isotopy, the vector fields X and Y induce an Anosov vector field Z on the manifold $\mathscr{M} = P \cup Q / \varphi$.

5.3. Periodic orbit complements as JSJ pieces of transitive Anosov flows. The following proposition allows us to realize periodic orbit complements of Anosov or pseudo-Anosov flows as JSJ pieces of transitive Anosov flows. We recall that a $\mathscr{C}^1$ flow X^t on a closed 3-manifold $\mathscr{M}$ is said to be pseudo-Anosov if it is locally modeled on a semi-branched covering of an Anosov flow [15]. In other word it is a generalization of an Anosov flow where we allow a finite number of singularities of stable and unstable foliations of p-prong type, $p \geq 3$.

Proposition 5.6. *Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a finite collection of periodic orbits of a transitive pseudo-Anosov vector field X on an orientable 3-manifold $\mathscr{M}$. Assume that all the singular orbits of X are contained in Γ and that the complementary $\mathscr{M} \smallsetminus \Gamma$ is atoroidal. Then there exists an orientable 3-manifold $\mathscr{N}$ carrying a transitive Anosov vector field Y such that the JSJ decomposition of $\mathscr{N}$ is made of two atoroidal pieces P and P', both homeomorphic to $\mathscr{M} \smallsetminus \Gamma$, and a periodic Seifert piece. The restriction of Y to P and P' is obtained from X by a DA*[5] *bifurcation on the orbits of Γ.*

A Seifert piece in a 3-manifold $\mathscr{M}$ carrying an Anosov vector field X is said to be *periodic* if there exists a Seifert fibration for which the regular fiber is homotopic to a power of a periodic orbit of the flow of X.

We refer to the proof of [22, Proposition 10.0.7]. The idea is to perform successively an *attractive and a repelling DA bifurcation* on each orbit $\gamma_1, \dots, \gamma_n$ of the collection Γ. We show the existence of solid tori T_i in the neighborhood of each γ_i, whose boundary is quasi-transverse to the modified vector field, and such that the complementary $\mathscr{M} \smallsetminus \bigcup_i T_i$ is a building block. We use Theorem 1 to glue this building block to one with a "matching" strongly quasi-transverse boundary lamination.

[5] Derived from Anosov

As a corollary of this proposition we obtain sufficient conditions for the complementary of a *hyperbolic knot* in $\mathbb{S}^3$ to be realized as an atoroidal JSJ piece of a transitive Anosov flow. Here is an example of such knots.

Example 5.7 ([22, Proposition 10.4.3]). Let $K = \partial(S_1 \# S_2 \# \ldots \# S_n)$ be a plumbing of n copies of the Seifert surface of the figure eight knot. Then K is a hyperbolic fibered knot and the complementary of K is an atoroidal JSJ piece of a manifold carrying a transitive Anosov flow.

5.4. Gluing pieces of skewed $\mathbb{R}$-covered Anosov flows. An Anosov flow on a 3-manifold $\mathcal{M}$ is said to be $\mathbb{R}$-*covered* if the leaf space of the lifted stable foliations $\widetilde{\mathcal{F}}^s$ on the universal cover $\widetilde{\mathcal{M}}$ is separated (hence homeomorphic to $\mathbb{R}$). It is said to be *skewed* $\mathbb{R}$-*covered* if it is moreover not orbitally equivalent to a suspension. The proposition below allows us to cut building blocks out of a skewed $\mathbb{R}$-covered Anosov flow along a collection of incompressible tori.

Proposition 5.8 ([3, Theorem A' and Theorem E])**.** *Let Z be a skewed $\mathbb{R}$-covered Anosov vector field on a closed orientable 3-manifold $\mathcal{M}$, whose stable and unstable foliations are transversely orientable. Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be a finite collection of incompressible tori embedded in $\mathcal{M}$, pairwise disjoint and pairwise non homotopic. Then there exists a collection $\mathcal{T}' = \{T'_1, \ldots, T'_n\}$ of pairwise disjoint tori isotopic to $\mathcal{T}$ and quasi-transverse to Z, and this collection is unique up to homotopy along the orbits of the flow. As a consequence, if we set $P := \mathcal{M} \smallsetminus \mathcal{T}'$, then $(P, Z\big|_P)$ is a building block.*

Definition 5.9 (Skewed $\mathbb{R}$-covered Anosov block)**.** In the setting of Proposition 5.8, we call a *skewed* $\mathbb{R}$-*covered Anosov block* any union of connected components of $(P, Z\big|_P)$.

Proposition 5.10. *Let (P, X) and (P', X') be two skewed $\mathbb{R}$-covered Anosov blocks and $\varphi\colon \partial P \to \partial P'$ a gluing map. There is a gluing map ψ isotopic to φ among gluing maps such that the vector field Z induced by X and X' on $P \cup P'/\psi$ is Anosov.*

We refer to the proof of [22, Proposition 11.0.2]. Let us add that the Anosov flow obtained by Proposition 5.10 is still $\mathbb{R}$-covered: this can easily be proved using knowledge of the orbit space of Anosov flows [2]. Note that this statement does not require any assumption on the action of the gluing map on the boundary laminations. This follows from the particular type of foliations induced by the stable and unstable foliations of Z on a quasi-transverse torus T embedded in a skewed $\mathbb{R}$-covered Anosov flow: they both contain no compact leaves other than the periodic orbits in the boundary, and are without Reeb components (see Figure 12). This is a consequence of a result of T. Barbot [3].

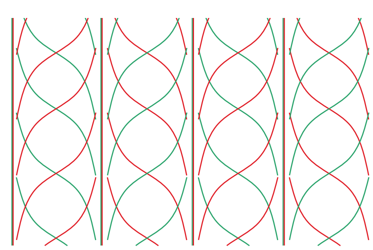

FIGURE 12. Trace of the stable and unstable foliation on a quasi-transverse torus embedded in a skewed $\mathbb{R}$-covered Anosov flow

Any *piece of geodesic flow* or finite covering of geodesic flow is a skewed $\mathbb{R}$-covered Anosov block. Hence Proposition 5.10 generalizes the Handel–Thurston construction, as well as recent generalization by A. Clay and T. Pinski [16] and T. Barbot and S. Fenley [9], and gets rid of the *positivity constraint* on the isotopy class of the gluing map. Note, however, that it allows us to use blocks that are much more general than geodesic flow pieces and that are not *a priori* cut out from the same Anosov flow. Recall that skewed $\mathbb{R}$-covered Anosov flows form a rich family of Anosov flows. S. Fenley showed in [17] that any Anosov flow obtained by Dehn–Goodman–Fried surgeries *of coherent orientations* on a suspension or a geodesic flow is skewed $\mathbb{R}$-covered. C. Bonatti and I. Iakovoglou showed in [13] that if X is an Anosov field obtained by Dehn–Goodman–Fried surgeries from a suspension then any surgery on an ϵ-dense periodic orbit of X yields a skewed $\mathbb{R}$-covered Anosov flow.

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