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Time change for unipotent flows and rigidity

Elon Lindenstrauss & Daren Wei

(Recommended by Boris Hasselblatt)

Abstract. We prove a dichotomy regarding the behavior of one-parameter unipotent flows on quotients of semisimple lie groups under time change. We show that if $u_t^{(1)}$ acting on G_1/Γ_1 is such a flow it satisfies exactly one of the following:

(1) The flow is loosely Kronecker, and hence isomorphic after an appropriate time change to any other loosely Kronecker system.
(2) The flow exhibits the following rigid behavior: if the one-parameter unipotent flow $u_t^{(1)}$ on G_1/Γ_1 is isomorphic after time change to another such flow $u_t^{(2)}$ on G_2/Γ_2, then G_1/Γ_1 is isomorphic to G_2/Γ_2 with the isomorphism taking $u_t^{(1)}$ to $u_t^{(2)}$ and moreover the time change is cohomologous to a trivial one.

The full details will appear in a later publication.

1. Introduction

Unipotent flows have some striking rigidity properties: for instance, every orbit is recurrent [10], and every orbit closure as well as every invariant measure is homogeneous [20, 21, 22].

In particular, they are known to be rigid with respect to measurable isomorphisms. To fix notations, for $i = 1,2$, let G_i be the identity component of a real semisimple linear algebraic group without compact factors, and let $\Gamma_i < G_i$ be a lattice. Let $\mathscr{B}_i$ be the Borel σ-algebra on G_i/Γ_i, let m_i on G_i/Γ_i be the normalized Haar measure, and suppose that $u_t^{(i)}$ are one parameter unipotent subgroups of G_i respectively. This means that $u_t^{(i)} = \exp(t\mathbf{u}^{(i)})$ with $\mathbf{u}^{(i)}$ a nilpotent element of the Lie algebra $\mathfrak{g}_i$ of G_i (considered as a matrix). In [17], Ratner showed that if $G_1 = G_2 = \mathrm{SL}_2(\mathbb{R})$ and $\psi : G_1/\Gamma_1 \to G_2/\Gamma_2$ is a measurable isomorphism between the flows $(G_1/\Gamma_1, \mathscr{B}_1, m_1, u_t^{(1)})$ and $(G_2/\Gamma_2, \mathscr{B}_2, m_2, u_t^{(2)})$, then ψ is automatically given by a very simple (and algebraic) form. Concretely, there is an automorphism of algebraic groups $\Psi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ and $C \in \mathrm{SL}_2(\mathbb{R})$ so that $\Gamma_2 = \Psi(\Gamma_1)$, $u_t^{(2)} = C\Psi(u_t^{(1)})C^{-1}$ so that ψ sends the coset $[g]_{\Gamma_1} \in G_1/\Gamma_1$ to $[C\Psi(g)]_{\Gamma_2}$. By measurable isomorphism we mean that ψ is one-to-one and onto between conull measurable subsets of G_i/Γ_i, sends m_1 to m_2 and intertwines with the corresponding $\mathbb{R}$-actions, i.e., for every $t \in \mathbb{R}$

$$\psi(u_t^{(1)}.x) = u_t^{(2)}.\psi(x) \qquad \mu_1\text{-a.e..}$$

This rigidity of measurable isomorphisms was generalized by Witte Morris [25, 26]. Rigidity of unipotent flows under measurable isomorphisms can also be deduced from the much stronger measure classification results of Ratner in [20, 21] (this is explicitly

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mentioned by Ratner in [20, Cor. 6]). We refer the reader to Witte Morris's book [12] for more details and background.

The isomorphism rigidity theorem tells us that a weaker notion of isomorphism between two flows in the class of unipotent flows on quotients of semisimple groups, say G_1/Γ_1 and G_2/Γ_2 implies a much stronger notion of isomorphism. Namely G_1 and G_2 are isogenous as algebraic groups, and if we chose G_1 and G_2 (as we may) so that G_i acts faithfully[1] on G_i/Γ_i, then in fact G_1 and G_2 need to be isomorphic with the isomorphism taking Γ_1 to Γ_2 and $u_t^{(1)}$ to a conjugate of $u_t^{(2)}$.

In this paper we allow a much weaker notion of equivalence between flows—known as **monotone equivalence** (we shall also use interchangeably the term **Kakutani equivalence**, though strictly speaking the latter term is most often used for a closely related notion for $\mathbb{Z}$-actions) and investigate if a similar rigidity holds. It turns out that for this equivalence condition a rather striking dichotomy holds: one parameter unipotent flows on quotients of semisimple groups fall into two categories:

- unipotent flows that are loosely Kronecker and hence are all monotone equivalent to each other;
- the non loosely Kronecker unipotent flows where (the weak) monotone equivalence implies the much stronger (algebraic) equivalence as above.

We stress that in the loosely Kronecker case monotone equivalence does **not** imply isomorphism as measure preserving flows. The full details will appear in a later publication. This reference also appeared at the very end of the abstract; we present here some of the ingredients as well as an outline of the argument. An explicit classification of which unipotent flows are loosely Kronecker was obtained by Kanigowski, Vinhage and the second named author in [5] (see below).

Let $(X_i,\mathscr{B}_i,\mu_i)$ for $i=1,2$ be two standard Borel probability measure spaces, and let $u_t^{(i)}: X_i \to X_i$ be an ergodic flow ($\mathbb{R}$-action) on X_i preserving m_i. The two flows $(X_i,\mathscr{B}_i,\mu_i,u_t^{(i)})$ are **monotone** (or **Kakutani**) **equivalent** if there is a one-to-one and onto measurable map $\psi: X_1' \to X_2'$ with $\mu_i(X_i \setminus X_i')=0$ ($X_i' \subseteq X_i$), and so that μ_2 is in the same measure class as $\psi_*\mu_1$, taking $u_\bullet^{(1)}$-orbits to $u_\bullet^{(2)}$-orbits **preserving the order structure**, i.e., so that if $x, u_t^{(1)}.x \in X_1'$ with $t>0$ then $\psi(u_t^{(1)}.x) = u_\tau^{(2)}.\psi(x)$ for $\tau>0$. The condition that ψ preserves the order structure on the orbits is essential, as any two ergodic flows will be orbit equivalent (by work of Ornstein and Weiss, this extends to actions of general amenable groups).

Given an ergodic measure preserving flow $(X,\mathscr{B},\mu,u_t)$ and a function $\alpha \in L_+^1(X,\mathscr{B},\mu)$ (one can think of $1/\alpha$ as the time change velocity) we can define a new flow on X, monotone equivalent to the original flow, via **time change**. This new flow $u_{\alpha,t}$ is given by

$$u_{\alpha,\tau}(x)=u_t(x) \qquad \text{where } \tau \text{ and } t \text{ satisfy } \int_0^t \alpha(u_s x)\,ds=\tau.$$

The new flow $u_{\alpha,t}$ preserves a measure μ^α in the same measure class as μ, with $d\mu^\alpha = \alpha d\mu/(\int \alpha d\mu)$.

Monotone equivalence can be characterized in terms of time change: $(X_1,\mathscr{B}_1,\mu,u_t^{(1)})$ is monotone equivalent to $(X_2,\mathscr{B}_2,\mu_2,u_t^{(2)})$ iff there is an $\alpha \in L_+^1(X_1,\mathscr{B}_1,\mu_1)$ so that the time changed system $(X_1,\mathscr{B}_1,\mu_1^\alpha,u_{\alpha,t}^{(1)})$ is measurably isomorphic to $(X_2,\mathscr{B}_2,\mu_2,u_t^{(2)})$. If $\int \alpha d\mu = 1$ we will say that ψ is an **even Kakutani equivalence**; this implies that

$$\psi(u_t^{(1)}x)=u_{t+o_x(t)}^{(2)}\psi(x) \qquad \text{a.s. as } t\to\infty.$$

[1] By replacing, if needed, the G_i with an appropriate quotient by a finite subgroup in the center of G_i.

A monotone equivalence $\psi : (X_1,\mathscr{B}_1,\mu,u_t^{(1)}) \to (X_2,\mathscr{B}_2,\mu_2,u_t^{(2)})$ gives rise to a **cocycle** $\tau(x,t)$ on $(X_1,\mathscr{B}_1,\mu,u_t^{(1)})$ defined by the relation $\psi(u_t^{(1)}.x) = u^{(2)}_{\tau(x,t)}.\psi(x)$. Recall that two cocycles τ,τ' on X_1 are **cohomologous** if there is a $w : X_1 \to \mathbb{R}$ so that

$$\tau'(x,t) = \tau(x,t) + w(u_t^{(1)}.x) - w(x).$$

Definition 1.1. Let ψ and ψ' be two Kakutani equivalences between $(X_1,\mathscr{B}_1,\mu,u_t^{(1)})$ and $(X_2,\mathscr{B}_2,\mu_2,u_t^{(2)})$ inducing cocycles τ,τ' on X_1 as above. We say ψ is **cohomologous** to ψ' if there is a $w : X_1 \to \mathbb{R}$

$$\psi'(x) = u^{(2)}_{w(x)} \circ \psi(x).$$

In particular $\tau'(x,t) = \tau(x,t) + w(u_t^{(1)}.x) - w(x)$ hence τ and τ' are cohomologous.

In the context we study in this paper, i.e. when both X_1 and X_2 are quotients of semisimple groups, and $u_t^{(1)}, u_t^{(2)}$ are one parameter unipotent groups, it is easy to "rectify" any monotone equivalence between these flows to an even Kakutani equivalence (see Proposition 3.1). Thus we may restrict ourselves to even Kakutani equivalences. The following is our main theorem:

Theorem 1.2. *For $i = 1,2$, let G_i be the identity component of a real semisimple linear algebraic group without compact factors, $\Gamma_i < G_i$ a lattice, m_i the probability measure on G_i/Γ_i induced by Haar measure on G_i, $u_t^{(i)}$ a one-parameter unipotent subgroup of G_i, and $\mathfrak{g}_i$ the Lie algebra of G_i. Assume for $i = 1,2$, the group $u_t^{(i)}$ acts ergodically on $(G_i/\Gamma_i, m_i)$ and that G_i acts faithfully on G_i/Γ_i. Then $(G_1/\Gamma_1,\ m_1,\ u_t^{(1)})$ and $(G_2/\Gamma_2,\ m_2,\ u_t^{(2)})$ are Kakutani equivalent if and only if one of the following holds:*

(A1) *for both $i = 1$ and $i = 2$, we have that $\mathfrak{g}_i = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{g}'_i$ and the generator of $u_t^{(i)}$ is of the form*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times 0 \in \mathfrak{g}_i;$$

(A2) *there exist an isomorphism $\phi : G_1 \to G_2$ and $C \in G_2$ such that $\phi(\Gamma_1) = \Gamma_2$ and $\phi(u_t^{(1)}) = Cu_t^{(2)}C^{-1}$. Moreover, any even Kakutani equivalence between these systems is cohomologous to an actual isomorphism.*

Recall that $(X,\mathscr{B},\mu,T_t)$ is **loosely Kronecker**[2] if T_t is Kakutani equivalent to a linear irrational flow on $\mathbb{T}^2$. Ratner in [14] showed that horocycle flows (i.e. the case of $G = \mathrm{SL}_2(\mathbb{R})$) are loosely Kronecker and thus no rigidity results can hold for L^1 time changes of horocycle flows. Kanigowski, Vinhage and the second named author [5, Theorem 1.1] showed that $(G/\Gamma, m, u_t)$ in fact is loosely Kronecker if and only if (A1) holds. On the other hand Ratner showed that if one assume the time change is done according to a function α that is Hölder in the $\mathrm{SO}_2(\mathbb{R})$-direction then this implies isomorphism [19]. Further rigidity statements under assumptions, e.g., like in [19] can be found in [24, 1].

On the other hand, Ratner showed in [15] that if $X_1 = \mathrm{SL}_2(\mathbb{R})/\Gamma_1 \times \mathrm{SL}_2(\mathbb{R})/\Gamma_1$ and $u_t^{(1)} = \left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & t \\ 0 & 1\end{smallmatrix}\right)\right)$ then $(X_1,\mathscr{B},m_1,u_t^{(1)})$ is not loosely Kronecker hence some rigidity holds—at the very least, the product of two horocycle flows is not monotone equivalent to a single horocycle flow. Later [16] Ratner defined an invariant under monotone equivalence she called the Kakutani invariant, and used it to show that a product of k horocycle flows is not isomorphic to a product of ℓ horocycle flows if $k \neq \ell$. Ratner's invariant was calculated in the generality we consider in this paper by Kanigowski, Vinhage and the second named author in [5]. In particular, Ratner's invariant turns out to

[2]Such systems are also known as zero entropy loosely Bernoulli or standard, though the latter term has different meanings in different contexts.

depend only on the group G_1 and the one-parameter unipotent group $u_t^{(1)}$, but not on the lattice $\Gamma_1 < G_1$.

In her ICM1994 proceedings paper [23, p. 179], Ratner asked the following questions:

(1) Does there exist an L^1-time change between product of horocycle flows on $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ with different lattices?
(2) Does the isomorphism rigidity hold for smooth time changes of unipotent flows?

Our main theorem, Theorem 1.2, clearly answers the first question, and moreover shows that except for the loosely Kronecker case **no smoothness assumption** on the time changes is needed. In particular:

Corollary 1.3. *Let G be an identity component of a real semisimple algebraic group, $\Gamma < G$ a lattice such that G acts faithfully on G/Γ, m the probability measure on G/Γ induced by Haar measure on G and u_t a one-parameter unipotent subgroup acting ergodically on $(G/\Gamma, m)$. Then $(G/\Gamma, m, u_t)$ has time change rigidity if and only it is not loosely Kronecker.*

In a recent paper, Gerber and Kunde [3] investigated the descriptive set theoretic complexity of the general monotone equivalence problem for flows, showing that in full generality one cannot classify measure preserving flows up to monotone equivalence using any single (or indeed, countably many) invariants. In their paper they ask about the special case of unipotent flows and Ratner's Kakutani invariant. Our result provides an answer for this question:

Corollary 1.4. *Let $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, $\Gamma_1, \Gamma_2 < G$ two non-conjugate lattices, $u_t^{(i)} = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right)$ and m_i the probability measure on G/Γ_i induced by Haar measure on G for $i = 1, 2$. Then $(G/\Gamma_1, m_1, u_t^{(1)})$ and $(G/\Gamma_2, m_2, u_t^{(2)})$ are not Kakutani equivalent while they have the same value of Kakutani invariant.*

2. Some preliminaries

In this section we present some notations that are needed to state the key steps in the proof of Theorem 1.2.

2.1. $(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \boldsymbol{R})$-two sides matchable. In analogy to Feldman's $\bar{f}$ metric we will make use of the notion of (ϵ, δ, R)-two sided matching between two points. For more details and background we refer the reader to [2, 6, 13, 16].

Let l denotes the Lebesgue measure on $\mathbb{R}$ and consider for $R > 1$ the orbit segment $I_R(x) = \{T_s x : s \in [-R, R]\}$.

Definition 2.1 (cf. [16, Def. 1])**.** Let T_t be a flow on $(X, \mathscr{B}, \mu)$ and d a metric on X. Suppose $\delta, \epsilon \in (0,1)$ and $R > 1$. We say $x, y \in X$ are **$(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \boldsymbol{R})$-two sides matchable** if there exist a subset $A = A_{(x,y)} \subset [-R, R]$ with $l(A) > (1-\epsilon)2R$, and an increasing absolutely continuous map $h = h_{(x,y)} : A \to [-R, R]$, such that

(1) $d(T_{h(t)} x, T_t y) < \delta$ for all $t \in A$;
(2) $h(0) = 0$ and $0 \in A$;
(3) $|h'(t) - 1| < \epsilon$ for all $t \in A$.

We call h an **$(\boldsymbol{\delta}, \boldsymbol{\epsilon}, \boldsymbol{R})$-two sided matching** between $I_R(x)$ to $I_R(y)$.

2.2. Our setup. In this paper, we take G to be the identity component of a real semisimple linear algebraic group. See, for example, [11, Ch. 1] for a concise introduction to the theory of such groups. Let $\mathfrak{g}$ denote the Lie algebra of G.

Let d_G denote a right invariant Riemannian metric on G, which also induces a Riemannian metric $d_{G/\Gamma}$ on G/Γ. Moreover, the corresponding Riemannian volume on G defines a Haar measure $\tilde{m}$ on G and thus induces a probability measure m on G/Γ.

2.3. $\mathfrak{sl}_2(\mathbb{R})$-triple, chain basis, and optimal matching function. In order to give a precise description of the divergence rate for nearby points in G/Γ, the following special basis in Lie algebra $\mathfrak{g}$ is very helpful. We make us of the standard notations about exponential map, adjoint action and their relations; we refer the reader to [9] for more details.

Let $\mathbf{u} \in \mathfrak{g}$ be a nilpotent element; by Jacobson–Morozov theorem [4], there exists a homomorphism $\varphi : \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{g}$ such that

$$\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \mathbf{u}, \quad \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \mathbf{a}, \quad \varphi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \mathbf{u}^-,$$

where $\mathbf{a} \in \mathfrak{g}$ is an $\mathbb{R}$-diagonalizable element and $\mathbf{u}^- \in \mathfrak{g}$ is a nilpotent element. We shall use the following three one-parameter subgroups:

- the one-parameter unipotent subgroup
$$U = \left\{u_t = \exp(t\mathbf{u}) : t \in \mathbb{R}\right\};$$
- the corresponding one-parameter diagonalizable subgroup
$$A = \left\{a_t = \exp(t\mathbf{a}) : t \in \mathbb{R}\right\};$$
- the "opposite" one-parameter subgroup
$$V = \left\{v_t = \exp(t\mathbf{u}^-) : t \in \mathbb{R}\right\}.$$

Let $\mathfrak{h} = \operatorname{span}\left\{\mathbf{u}, \mathbf{a}, \mathbf{u}^-\right\}$ and let $H < G$ be the connected Lie subgroup (in the Hausdorff topology) generated by $\exp(\mathfrak{h})$.

By the well-known classification of finite dimensional Lie representations of $\mathfrak{sl}_2(\mathbb{R})$, we can find a basis of $\mathfrak{g}$ of the form

$$\left\{\mathbf{u}, \mathbf{a}, \mathbf{u}^-, \mathbf{x}^{0,1}, \dots, \mathbf{x}^{m_1,1}, \dots, \mathbf{x}^{0,n}, \dots, \mathbf{x}^{m_n,n}\right\}, \tag{2.1}$$

where for each $1 \le j \le n$, we have that $\mathbf{x}^{0,j}, \dots, \mathbf{x}^{m_j,j}$ is an irreducible representation of $\mathfrak{h}$ under ad with $\mathrm{ad}_{\mathbf{u}} \mathbf{x}^{i,j} = \mathbf{x}_{i+1,j}$. We call this basis the chain basis for $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Given a subset $W \subset \mathfrak{g}$, we define $C_{\mathfrak{g}}(W)$ to be the common centralizer of all $\mathbf{w} \in W$, i.e.,

$$C_{\mathfrak{g}}(W) = \left\{\mathbf{y} \in \mathfrak{g} : \mathrm{ad}_{\mathbf{w}}(\mathbf{y}) = 0 \ \forall\, \mathbf{w} \in W\right\}. \tag{2.2}$$

In order to find the best matching between the U-orbits of two nearby points, say $u_t.x$, and $u_t.wx$ one needs to modify the element of U used in one of these two points to compensate for the shearing behaviour of the unipotent flow. Explicitly, we have the following:

Lemma 2.2. *Suppose*

$$w = \exp(\vartheta_{\mathbf{u}^-}\mathbf{u}^-)\exp(\vartheta_{\mathbf{a}}\mathbf{a}) \in G.$$

Then there exists a rational function $\phi(t)$ (that depends on w) such that

$$\exp(\phi(t)\mathbf{u})\, w \exp(-t\mathbf{u}) = \exp(\vartheta_{\mathbf{u}^-,t}\mathbf{u}^-)\exp(\vartheta_{\mathbf{a},t}\mathbf{a}), \tag{2.3}$$

so that if $|t| < \epsilon\left|\vartheta_{\mathbf{u}^-}^{-1}\right|$ and $|\vartheta_{\mathbf{a}}| < \epsilon$ then

$$\left|\vartheta_{\mathbf{a},t}\right| < 2\epsilon, \quad \left|\vartheta_{\mathbf{u}^-,t}\right| < 2|\vartheta_{\mathbf{u}^-}|.$$

Moreover, $\left|\phi'(t) - 1\right| < \sqrt{\epsilon}$.

This lemma can be viewed as a special case of what Ratner calls the H-property (see [18, Def. 1]). Equation (2.3) and the following inequalities can be obtained by direct computation of 2×2 matrices. Cf. [5, Lemma 5.3] for more details.

2.4. Kakutani–Bowen ball. We can represent elements $g \in G$ sufficiently close to identity using the following local chart based on our choice of basis for $\mathfrak{g}$ in (2.1):

$$g = \exp(\vartheta_{\mathbf{u}}(g)\,\mathbf{u})\exp(\vartheta_{\mathbf{u}^-}(g)\,\mathbf{u}^-)\exp(\vartheta_{\mathbf{a}}(g)\,\mathbf{a})\breve{g}, \tag{2.4}$$

where

$$\breve{g} = \exp\left(\sum_{j=1}^{n}\sum_{i=0}^{m_j}\vartheta_{i,j}(g)\,\mathbf{x}^{i,j}\right).$$

Definition 2.3.

(1) For $\epsilon > 0$ and $R > 1$, let $\mathrm{Bow}(R,\epsilon)$ be the set

$$\left\{g \in G : d_G(\exp(t\mathbf{u})g\exp(-t\mathbf{u}), e) < \epsilon\ \forall\, t \in [-R,R]\right\},$$

and define for $x \in G/\Gamma$ the **(R,ϵ)-Bowen ball** around x to be

$$\mathrm{Bow}(R,\epsilon,x) = \mathrm{Bow}(R,\epsilon).x.$$

(2) For $\epsilon > 0$ and $R > 1$, let $\mathrm{Kak}(R,\epsilon)$ be the set of $g \in G$ satisfying the following:
   (a) $\left|\vartheta_{\mathbf{u}^-}(g)\right| < \frac{\epsilon}{R}$,
   (b) $\left|\vartheta_{\mathbf{a}}(g)\right| < \epsilon$,
   (c) $\left|\vartheta_{\mathbf{u}}(g)\right| < \epsilon$,
   (d) $\breve{g} \in \mathrm{Bow}(R,\epsilon)$,

where $\vartheta_{\mathbf{u}^-}$, $\vartheta_{\mathbf{a}}$, $\vartheta_{\mathbf{u}}$ and $\breve{g}$ as in (2.4). For $x \in G/\Gamma$, we define the **(R,ϵ)-Kakutani–Bowen ball** around it to be

$$\mathrm{Kak}(R,\epsilon,x) = \mathrm{Kak}(R,\epsilon).x,$$

and similarly set $\mathrm{Kak}(R,\epsilon,\tilde{x}) = \mathrm{Kak}(R,\epsilon).\tilde{x}$ for $\tilde{x} \in G$.

It is easy to see using the "time change" given in Lemma 2.2 that if $x \in G/\Gamma$ and $y \in \mathrm{Kak}(R,\epsilon,x)$ then x and y are $(C\epsilon,\sqrt{\epsilon},R)$-matchable for an appropriate constant C, and similarly for $\tilde{x},\tilde{y} \in G$. On G, the converse also essentially holds, and moreover the best matching among all two sided matching is essentially given by the matching defined in §2.1. This turns out to be also true in G/Γ, but this is a much more delicate statement and only holds **if** the action of u_t on G/Γ is not loosely Kronecker—indeed, this fact is essentially equivalent to our Main Lemma presented in §4.

Proposition 2.4. *There exist constants $C_1 > 0$ and L such that for any $\epsilon,\delta > 0$ small enough the following holds. Suppose $\tilde{x} \in G$, $d_G(g,e) < \delta$ and $\tilde{y} = g.\tilde{x}$. Let $h : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $h(0) = 0$ and $\left|h'(t) - 1\right| < \epsilon$. Then we can cover the set*

$$I_{\tilde{x},\tilde{y}} = \left\{t \geq 0 : d_G(\exp(h(t)\mathbf{u})\tilde{x}, \exp(t\mathbf{u})\tilde{y}) < 4\delta\right\},$$

by $k \leq L$ disjoint closed intervals $[b_i,d_i]$, so that on each interval

$$d_G(\exp((\phi(t) - \phi(b_i) + h(b_i))\mathbf{u})\tilde{x}, \exp(t\mathbf{u})\tilde{y}) < C_1\delta \quad \forall\, t \in [b_i,d_i]$$

with $\phi(t)$ is as in Lemma 2.2. Moreover for all $i = 1,\dots,k$ and $t \in [b_i,d_i]$,

$$\exp((\phi(t) - \phi(b_i) + h(b_i))\mathbf{u})\tilde{x} \in \mathrm{Kak}(d_i - b_i, C_1\delta, \exp(t\mathbf{u})\tilde{y}).$$

2.5. Two relations. Let $K \subset G/\Gamma$ be a compact set, $x, y \in K$ and ϵ small enough so that for any $x \in K$, the map $g \mapsto g.x$ is injective on the ϵ-ball around $e \in G$. Suppose that $y = g.x$ for $g \in G$ with $d_G(g,e) < \epsilon$ and moreover that $d_{G/\Gamma}(u_t.x, u_s.y) < \epsilon$ for some $t, s \geq 0$. Lift x to a point $\tilde{x}$ in G and set $\tilde{y} = g\tilde{x}$, so that $d_G(\tilde{x},\tilde{y}) < \epsilon$. Then there exists a unique $\gamma \in \Gamma$ such that $d_G(\exp(t\mathbf{u})\tilde{x}, \exp(s\mathbf{u})\tilde{y}\gamma) < \epsilon$. Note that the conjugacy class $[\gamma]$ of γ does not depend on the lift $\tilde{x}$ we choose for x. If $x, y, s, t, [\gamma]$ satisfy the above we shall write

$$(x,y) \overset{[\gamma]}{\rightsquigarrow} (u_t.x, u_s.y);$$

we also write $(x,y) \overset{e}{\rightsquigarrow} (u_t.x, u_s.y)$ for $(x,y) \overset{[e]}{\rightsquigarrow} (u_t.x, u_s.y)$ and

$$(x,y) \overset{\neq e}{\rightsquigarrow} (u_t.x, u_s.y)$$

if $(x,y) \overset{[\gamma]}{\rightsquigarrow} (u_t.x, u_s.y)$ for $\gamma \neq e$.

3. Reduction to a good Kakutani equivalence

We begin with two reductions for any general Kakutani equivalence between the actions of unipotent one-parameter subgroups. The second step of these reductions also holds for any general abstract ergodic systems.

For $i = 1,2$, we assume that G_i is the identity component of a real semisimple linear algebraic group without compact factors, $\Gamma_i < G_i$ a lattice and m_i a probability measure on G_i/Γ_i induced from Haar measure on G_i. Let $u_t^{(i)}$ be a one-parameter unipotent subgroup acting ergodically on $(G_i/\Gamma_i, m_i)$ for $i = 1,2$ and ψ an arbitrary Kakutani equivalence between $(G_1/\Gamma_1, u_t^{(1)}, m_1)$ and $(G_2/\Gamma_2, u_t^{(2)}, m_2)$.

The first step of the reduction is using the diagonal subgroup to normalize our one-parameter unipotent subgroup, which reduces ψ to an even Kakutani equivalence:

Proposition 3.1. *There exists $s_0 \in \mathbb{R}$ such that $a_{s_0}^{(2)} \circ \psi$ is an even Kakutani equivalence, where $a_s^{(2)}$ is the diagonalizable subgroup arising from $\mathfrak{sl}_2(\mathbb{R})$-triple for the generator of $u_t^{(2)}$ in §2.3.*

The target of the second step of our reduction is to obtain a good control of the time change function α. In order to simplify the notation, we introduce the following definition:

Definition 3.2. Let T_t and S_t be two ergodic flows acting on $(X,\mathscr{B},\mu)$ and $(Y,\mathscr{C},\nu)$ respectively. For any $\epsilon > 0$, we say ψ is an **ϵ-controllable Kakutani equivalence** between T_t and S_t if the following holds:

1. ψ is an even Kakutani equivalence between T_t and S_t with time change function α;
2. $\operatorname{esssup}|\alpha - 1| < \epsilon$;
3. there exists a full measure set $K \subset X$ such that for every $x \in K$, $\alpha(T_t x)$ is a C^∞ function in t.

Lemma 3.3. *Let ψ be an even Kakutani equivalence between two ergodic flows. For any given $\epsilon > 0$, there exists an ϵ-controllable Kakutani equivalence $\tilde{\psi}$ that is cohomologous to ψ.*

This lemma (which is a lemma about general ergodic systems and does not use any special properties of unipotent flows) follows from a combination of [6, Proposition 2.3] and [13, Theorem 1.4].

As a result of these two reductions, it is permissible to assume that ψ is an ϵ_1-controllable Kakutani equivalence, with $\epsilon_1 > 0$ a small fixed constant, an assumption we make from this point on.

4. Main lemma

One of the key ingredients of the proof of Theorem 1.2 is the following proposition, which essentially shows that Kakutani–Bowen balls are preserved under Kakutani equivalence map. This type of proposition first appeared in [15] for Cartesian products of horocycle flows on compact quotients of $\mathrm{SL}_2(\mathbb{R})$. In order to calculate an invariant for Kakutani equivalences introduced by Ratner [16] for more general one-parameter unipotent flows, Kanigowski, Vinhage and the second named author extended this analysis to general semisimple linear Lie groups in [5], though for our purposes we need a sharper form of this important result.

Lemma 4.1 (Main Lemma)**.** *There exist $\delta_1, R_1 > 0$ and a compact set $K_1 \subset G_1/\Gamma_1$ with $m_1(K_1) > 0.99$ such that for any $\delta \in (0, \delta_1)$, there exists $\epsilon \in (0, \delta)$ such that if $x, y \in K_1$ and $R > R_1$ satisfy*

$$x \in \mathrm{Kak}(R, \epsilon, u^{(1)}_{t_0}.y)$$

for some $|t_0| < \epsilon R$, then we have

$$\psi(u^{(1)}_{t_0}.y) \in \mathrm{Kak}(R, \delta, u^{(2)}_{t_R}.\psi(x))$$

for some t_R satisfying $|t_R| \le 40\epsilon_1 R$.

This main lemma is at the heart of our whole argument as it provides an essential tool to establish that images of two nearby points in a good set under a Kakutani equivalence remain close; a key point is that for this purpose closeness needs to be measured in terms of Kakutani–Bowen balls.

In order to establish the main lemma, we first note that as in §2.4, $x \in \mathrm{Kak}(R, \epsilon, u^{(1)}_{t_0}.y)$ implies that x and $u^{(1)}_{t_0}.y$ are $(C\epsilon, \sqrt{\epsilon}, R)$-two sides matchable for some appropriate constant C. On a set of large measure, ψ is continuous, so combining the pointwise ergodic theorem with our assumption that ψ is an ϵ_1-controllable Kakutani equivalence (cf. the end of the previous section) give that if x and y are in a "good" set of large measure and R large enough then $\psi(x)$ and $\psi(u^{(1)}_{t_0}.y)$ are $(\delta, 2\epsilon_1, R)$-two sides matchable for some appropriate constant δ.

In view of this, Lemma 4.1 can be reduced to the following lemma, which establishes the connection between two sided matching in G_2/Γ_2 and Kakutani–Bowen balls.

Lemma 4.2. *There is a compact subset $K_2 \subset G_2/\Gamma_2$ with $m_2(K_2) \ge 0.99$ and constants C_2, L such that for any ϵ, δ sufficiently small and R sufficiently big the following holds. Let $x \in G_2/\Gamma_2$ and $y \in K_2$ be $(C_2\delta, \epsilon, R)$-two sides matchable with matching function h. Then there exist lifts $\tilde{x}, \tilde{y} \in G_2$ of x, y, $\gamma_R \in \Gamma_2$, $|s_R| \le R$ and $R' \in \left[\frac{R}{L}, R\right]$ such that for both $t = 0$ and R',*

$$\exp((h(s_R + t))\,\mathbf{u}_2)\tilde{x} \in \mathrm{Kak}(R', \delta)\exp((s_R + t)\,\mathbf{u}_2)\tilde{y}\gamma_R.$$

Roughly speaking, the above lemma says that if two points are two sides matchable for sufficiently long time, then they also stay close in the universal cover for a long time of the same order of magnitude.

Lemma 4.2 can be derived from the following lemma, where relation $\overset{\neq e}{\rightsquigarrow}$ is defined in §2.5. In order to simplify the notation, we define $x_t = u^{(2)}_t.x$ and $y_t = u^{(2)}_t.y$ for $x, y \in G_2/\Gamma_2$.

Lemma 4.3. *There exist $C_3, w, R_2 > 0$ and a compact set $K_3 \subset G_2/\Gamma_2$ with $m_2(K_3) > 0.99$ such that for every $R \ge R_2$ the following holds. Suppose that $y \in K_3$, $x \in \mathrm{Kak}(R, \delta, y)$, $x_{h(s)} \in \mathrm{Kak}(R, \delta, y_s)$ and $(x, y) \overset{\neq e}{\rightsquigarrow} (x_{h(s)}, y_s)$, then we have,*

$$\min(s, h(s)) \ge C_3 R^{1+w}.$$

Such a result first appeared in Ratner [15] for the Cartesian product of horocycle flows. By using techniques in algebraic group theory, we generalize it to arbitrary one-parameter unipotent subgroups. Together with some combinatorial estimates, Lemma 4.3 shows that for most points, the two sided matching on G_2/Γ_2 implies the two sided matching in the same time scale on G_2 for their corresponding lifts. This will give the proof of Lemma 4.2 and hence also Lemma 4.1.

5. Compatibility of diagonalizable group

For any ϵ,δ, if η is sufficiently small, $x \in G_1/\Gamma_1$ then x and $a^{(1)}_\eta x$ are (δ,ϵ,R)-two sides matchable for any R. Using Lemma 4.1 (the "Main Lemma") this implies that $\psi(a^{(1)}_\eta x) \in N_{G_2}(U^{(2)})\psi(x)$ with $N_{G_2}(U^{(2)})$ denoting the normalizer in G_2 of $U^{(2)}$, where $U^{(2)}$ is the one-parameter unipotent subgroup defined in §2.3. This can further be upgraded to the following statement:

Proposition 5.1. *There is an element c in the centralizer of $U^{(2)}$ so that if $\psi'(x) = c\psi(x)$ then a.e.*

$$\psi'(a^{(1)}_1.x) \in U^{(2)} a^{(2)}_1.\psi'(x).$$

Without loss of generality we may assume $\psi' = \psi$, and then Proposition 5.1 says that there is a measurable function $t(x) : X_1 \to \mathbb{R}$ so that for a.e. x,

$$\psi(a^{(1)}_1.x) = u^{(2)}_{t(x)} a^{(2)}_1.\psi(x). \tag{5.1}$$

Iterating, we can get

$$\psi(a^{(1)}_n.x) = u^{(2)}_{e^{2n-2}t(x)+e^{2n-4}t(a^{(1)}_1.x)+\cdots+t(a^{(1)}_{n-1}.x)} a^{(2)}_n.\psi(x). \tag{5.2}$$

Unfortunately, we have no information on $t(x)$ (we certainly do **not** know it is in L^1) so $e^{2n-2}t(x) + e^{2n-4}t(a^{(1)}_1.x) + \cdots + t(a^{(1)}_{n-1}.x)$ could be very large.

We overcome this significant hurdle by perturbing the points $a^{(1)}_j.x$ into a set where $t(\cdot)$ is bounded, using a certain L^2 ergodic theorem for $SL_2(\mathbb{R})$-actions described in the next section.

In her paper [19], Ratner used a statement analogous to Proposition 5.1 to show that **if the time change is sufficiently smooth**, a Kakutani equivalence between two horocycle flows has to come from an isomorphism. We follow a similar strategy here, using equation (5.1) as a crucial ingredient. To do that, we consider on the "conjugation" of ψ under the diagonalizable groups $a^{(i)}_t$

$$\psi_n(x) = a^{(2)}_{-n}\psi(a^{(1)}_n.x), \quad x \in G_1/\Gamma_1. \tag{5.3}$$

Combining the even Kakutani equivalence assumption and pointwise ergodic theorem, the limit of ψ_n (**if** it exists) will be a measurable isomorphism between the corresponding unipotent flows and thus the only remaining task is the existence of the limit of ψ_n. In [19], the regularity assumption of the time changes is used to establish the existence of the limit of ψ_n along a subsequence, thus establishing time change rigidity in this case were the time change is assumed to have some a priori regularity. A similar strategy has also been used very recently by Artigiani, Flaminio and Ravotti [1], however they need to **assume** convergence of the ψ_n to get an isomorphism. Due to the lack of regularity assumption in our situations, significant changes are needed.

6. Spectral gap and an $SL_2(\mathbb{R})$-ergodic theorem

The aim of this section is to state a $SL_2(\mathbb{R})$-pointwise ergodic theorem, which enable us to find good points in pushforward of small ϵ-balls in H_1 under $a^{(1)}_n$. We remark that the spectral gap for $L^2(G_1/\Gamma_1)$ is only used in this section to obtain the necessary pointwise behaviour.

Our $SL_2(\mathbb{R})$-ergodic theorem is following:

Theorem 6.1. *Let ρ be a unitary representation of $H = SL_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}_\rho$ with no fixed vectors and a spectral gap. Then there exists $\epsilon_2 > 0$ such that for any $\epsilon \in (0,\epsilon_2)$*

there is a $C(\epsilon)$ *so that*

$$\sum_{n=1}^{+\infty}\left\|\frac{1}{m_H(B_\epsilon^{H,\|\cdot\|})}\int_{B_\epsilon^{H,\|\cdot\|}}\rho(ha_{-n})v\,dm_H(h)\right\|^2 < C(\epsilon)\|v\|^2$$

for any $v\in\mathscr{H}_\rho$*, where* m_H *is the Haar measure on* H *and* $B_\epsilon^{H,\|\cdot\|}$ *is the* ϵ*-ball in* H *around identity with respect to the Hilbert–Schmidt norm.*

Recall that a unitary representation $(\rho,\mathscr{H})$ of H is said to have a spectral gap if there is some compactly supported probability measure ν on H so that on the orthogonal complement of the H-fixed vectors the norm of the operator $\int\rho(g)d\nu(g)$ is <1.

A direct corollary of Theorem 6.1 is:

Corollary 6.2. *Let* G *be the identity component of a real semisimple algebraic group without compact factors,* $\Gamma<G$ *a lattice with* m *the probability measure on* G/Γ *induced by Haar measure on* G*. Let* $H<G$ *be locally isomorphic to* SL_2 *and let* $a_t\in H$ *be a one-parameter diagonalizable subgroup as in §2.3. Given* $\epsilon\in(0,\epsilon_2)$ *and* $\eta\in[0,1)$*, then for any* $f\in L^2(G/\Gamma,m)$*,* m*-a.e.* $x\in G/\Gamma$*, we have*

$$\lim_{n\to+\infty}\frac{1}{m_H(B_\epsilon^{H,\|\cdot\|})}\int_{B_\epsilon^{H,\|\cdot\|}}f(a_{n(1-\eta)}ha_{n\eta}.x)dm_H(h)=\int_{G/\Gamma}f\,dm.$$

The deduction of Corollary 6.2 from Theorem 6.1 uses the fact that if G is a semisimple group without compact factors, Γ an irreducible lattice, and $H<G$ is as above, then the representation of H on $L^2(G/\Gamma)$ arising from left translations has a spectral gap. More generally, if $G=\prod_i G_i$ and $\Gamma=\prod_i\Gamma_i$ with $\Gamma_i<G_i$ irreducible lattices, and if H projects non-trivially into each G_i, then the action of H on $L^2(G/\Gamma)$ has a spectral gap. See, for example, [8, 7] for more information and background.

7. Existence of limiting map

Finally we show that $\psi_n(x)$ converges in an appropriate sense to a limiting map $\varphi(x)$. This limiting map will turn out to be not just an (even) Kakutani equivalence but in fact a measurable isomorphism between the two unipotent flows given by the $u_t^{(i)}$ action on G_i/Γ_i.

Theorem 7.1. *For* m_1*-a.e.* $x\in G_1/\Gamma_1$*, there is a subsequence* $\{n_i\}_{i\in\mathbb{N}}$ *of natural numbers with full density such that* $\varphi(x)=\lim_{i\to\infty}\psi_{n_i}(x)$ *exists and*

- *(B1)* $\varphi:G_1/\Gamma_1\to G_2/\Gamma_2$ *is a measurable map;*
- *(B2)* $\varphi(x)\in U^{(2)}(\psi(x))$*;*
- *(B3)* $\varphi(u_t^{(1)}.x)=u_t^{(2)}.\varphi(x)$ *for* m_1*-a.e.* $x\in G_1/\Gamma_1$ *and any* $t\in\mathbb{R}$*;*
- *(B4)* $\varphi(a_k^{(1)}.x)=a_k^{(2)}.\varphi(x)$ *for* m_1*-a.e.* $x\in G_1/\Gamma_1$ *and any* $k\in\mathbb{N}$*.*

The challenging part is establishing that there is a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of natural numbers with full density such that

$$\varphi(x)=\lim_{i\to\infty}\psi_{n_i}(x)$$

exists; the rest follows by standard arguments.

The key observation to prove Theorem 7.1 is to consider the iterations $\{a_n^{(1)}h_n.x\}_{n\in\mathbb{N}}$ for some $h_n\in B_\epsilon^{H_1,\|\cdot\|}$ instead of $\{a_n^{(1)}.x\}_{n\in\mathbb{N}}$. By using Corollary 6.2, we can always guarantee that $a_n^{(1)}h_n.x$ stays in some good sets, i.e., $|t(a_n^{(1)}h_n.x)|$ is bounded by a positive constant, for every n. Note that this was not possible for $a_n^{(1)}x$ due to the existence of bad iterations as guaranteed by pointwise ergodic theorem. In order to control the error terms in opposite horocycle directions, i.e., $\exp(\mathbb{R}\mathbf{u}_2^-)$, we also need to consider iterations $\{a^{(1)}_{n(1-\eta)} h_n a^{(1)}_{n\eta}.x\}_{n\in\mathbb{N}}$. With the help of estimates in representation theory and Lemma 4.1, we are able to combine two iterations and obtain controls for all directions.

Once we obtain Theorem 7.1, the proof of Theorem 1.2 follows from the combination of isomorphism rigidity theorem for unipotent flows and Ratner's argument in [19]. More precisely, (B1), (B3) together with the isomorphism rigidity theorem for unipotent flows give that G_1 and G_2 are isomorphic and that Γ_1 and Γ_2 are conjugate up to group isomorphism. Combining (B2), (B4), and Ratner's argument in [19, Proof of Theorem 1], we obtain that ψ indeed is cohomologous to a group isomorphism. This completes the proof.

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Elon Lindenstrauss: Einstein Institute of Mathematics, Edmond J. Safra Campus, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, 9190401, Israel
E-mail: elon@math.huji.ac.il

Daren Wei: Department of Mathematics, National University of Singapore, 119076, Singapore
E-mail: darenwei@nus.edu.sg